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REVIEW

Spontaneous Symmetry Breaking and Coherence in Two-Dimensional Electron–Hole and Exciton Systems

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The spontaneous breaking of the continuous symmetries of the two-dimensional (2D) electron–hole systems in a strong perpendicular magnetic field leads to the formation of new ground states and determines the energy spectra of the collective elementary excitations appearing over these ground states. In this review the main attention is given to the electron–hole systems forming coplanar magnetoexcitons in the Bose-Einstein condensation (BEC) ground state with the wave vector $\vec{k} = 0$, taking into account the excited Landau levels, when the exciton-type elementary excitations coexist with the plasmon-type oscillations. At the same time properties of the two-dimensional electron gas (2DEG) spatially separated as in the case of double quantum wells (DQWs) from the 2D hole gas under conditions of the fractional quantum Hall effect (FQHE) are of great interest because they can influence the quantum states of the coplanar magnetoexcitons when the distance between the DQW layers diminishes. We also consider in this review the bilayer electron systems under conditions of the FQHE with the one half filling factor for each layer and with the total filling factor for two layers equal to unity because the coherence between the electron states in two layers is equivalent to the formation of the quantum Hall excitons (QHExs) in a coherent macroscopic state. This makes it possible to compare the energy spectrum of the collective elementary excitations of the Bose-Einstein condensed QHExs and coplanar magnetoexcitons. The breaking of the global gauge symmetry as well as of the continuous rotational symmetries leads to the formation of the gapless Nambu-Goldstone (NG) modes while the breaking of the local gauge symmetry gives rise to the Higgs phenomenon characterized by the gapped branches of the energy spectrum. These phenomena are equivalent to the emergence of massless and of massive particles, correspondingly, in the relativistic physics. The application of the Nielsen-Chadha theorem establishing the number of the NG modes depending of the number of the broken symmetry operators and the elucidation when the quasi-NG modes appear are demonstrated using as an example related with the BEC of spinor atoms in an optical trap. They have the final aim to better understand the results obtained in the case of the coplanar Bose-Einstein condensed magnetoexcitons. The Higgs phenomenon results in the emergence of the composite particles under the conditions of the FQHE. Their description in terms of the Ginzburg-Landau theory is remembered. The formation of the high density 2D magnetoexcitons and magnetoexciton-polaritons with point quantum vortices attached is suggested. The conditions in which the spontaneous coherence could appear in a system of indirect excitons in a double quantum well structures are discussed. The experimental attempts to achieve these conditions, the main results and the accumulated knowledge are reviewed.

Keywords: Spontaneous Coherence, Electron–Hole Pair, Bose-Einstein Condensation, Exciton, Elementary Excitations, Symmetry Breaking.

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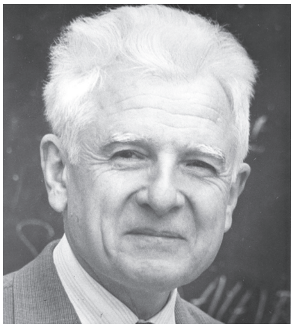
1. INTRODUCTION

The collective elementary excitations of the two-dimensional (2D) electron–hole (e–h) systems in a strong perpendicular magnetic field are discussed in the frame of the Bogoliubov theory of quasiaverages[1] taking into account the phenomena related with the spontaneous breaking of the continuous symmetries. The main results in this field have been obtained thanks to the fundamental papers by Goldstone,[2] Nambu,[3] Higgs[4] and Weinberg.[5]



S. A. Moskalenko was born on September 26, 1928 in the village Bravicha Calarash district, Republic of Moldova. Citizenship of Moldova. Moskalenko S. A. has graduated the Kishinev State University in 1951. In 1956–1959 he was post-graduate student of the Institute of Physics of the Academy of Sciences of Ukrainian SSR. He became Candidate of Physico-mathematical Sciences equivalent to PHD degree in 1961. Doctor of Physico-mathematical Sciences and the professor in the field of theoretical and mathematical physics beginning with 1971 and 1974 correspondingly. He became Laureate of the State prize in the field of Sciences and Technics of MSSR and USSR in 1981 and 1988 correspondingly. He was elected member-correspondent and full member of the Academy of Sciences of Moldova in 1989 and 1992 correspondingly. The concept of excitonic molecule was introdused. Later the exitonic molecule was called the biexciton. The biexciton represents the bound state of four Fermi quasiparticles (quaternions), namely, two electrons and two holes. More simple it can be regarded as a bound state of two excitons. The possibility of Bose-Einstein condensation (BEC) of quasiparticles with finite lifetime such as excitons and biexcitons was suggested. It was pointed out that BEC can occur in the quasiequilibrium conditions, when the relaxation time due to interpaticle scattering is much less than their life time and interaction between excitons is repulsive. The superfluidity of excitons and biexcitons can provide a new way of non-dissipative energy transfer in crystals. BEC can be induced by the resonant monochromatic photons. The results concerning the Bose-Einstein condensation of excitons and biexcitons as well the coherent nonlinear optics with excitons were reviewed in the monograph written together with Professor D. W. Snoke from Pittsburg University. Due to the collaboration with Professor M. A. Liberman from Uppsala University last years the properties of excitons in a strong magnetic field are studied. The polarizability, correlation energy and the dielectric liquid phase of Bose-Einstein condensed 2D magnetoexcitons with motional dipole moments were studied. The possible existence of the metastable dielectric liquid phase formed by Bose-Einstein condensed magnetoexcitons with wave vectors and motional dipole moments different from zero was established theoretically.

M. A. Liberman was born in Moscow, USSR on October 23, 1942. He graduated from Moscow State University in 1966. From 1969 to 2003 he worked at P. Kapitsa Institute for Physical Problems, Academy of Sciences USSR. He received his Ph.D. in 1971 from P. Lebedev Physical Institute in Moscow for the group theory in quantum mechanics and invariant expansion of the relativistic amplitudes, and then his Doctor of Physical and Mathematical Sciences degree in 1981 for a thesis on ionizing shock waves. Since 1991, he is professor of theoretical statistical physics working at the Physics Department, Uppsala University, Sweden. He is a citizen of both Russia and the Sweden. Among his achievements are the nonlinear theory of electromagnetic wave propagating in nonequilibrium plasmas (for example, in the ionosphere); a theory of the ionizing shock waves, exact solution for a hydrogen atom in a magnetic field of arbitrary strength, theory of a hydrogen molecule in a strong magnetic field, non-stationary nonlinear equation for a curved flame, theory of type I(a) supernova explosion. At Uppsala University he continue his work on combustion theory, for which he was recently nominated for Gold Medal of Combustion Institute, and he is also focused on the research of the Bose-Einstein condensate of excitons in low dimensional semiconductors in a strong magnetic field. He is author of the books: "Physics of Shock Waves in Gases and Plasmas," Springer-Verlag, 1985 (with A. Velikovich), "Physics of High-Density Z-pinch Plasmas," Springer-Verlag, 1998 (with J. DeGroot, A. Toor, R. Spielman), "Introduction to Physics and Chemistry of Combustion," Springer-Verlag, 2008.

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E. S. Moskalenko originally from Republic of Moldova was born in Chisinau in May 1963 and died unexpectedly in March 2012 in Sankt-Petersburg, Russia being 49 years old. He graduated the middle school in Chisinau in 1980 and the physical faculty of the Sankt-Petersburg State University in 1986. He was a doctoral student of the A. F. Ioffe Physico-Technical Institute (PTI) in Sankt-Petersburg of the Russian Academy of Sciences under the scientific guidance of the academician A. A. Kaplyanskii and in 1991 performed his doctoral thesis and obtained the candidatus scientiarum degree. His thesis was dedicated to the influence of the nonequilibrium acoustical phonons arising during the relaxation of the photo-created carriers on the photoluminescence of excitons in semiconductors Si and Cu2O. The exciton drag driven by the ballistic acoustical phonons was observed and the deficit of the low-frequency terahertz phonon in comparison with the Planck distribution function was revealed. The following 20 years of his post-doctoral activity were connected with the Department of the Solid State Optics of PTI, where he became a senior scientific collaborator growing as an experimental physicist in the range of optical spectroscopy of the solid states. His scientific interests were concentrated in the physics of excitons in semiconductors at high level of excitations and low temperatures. In the first decade of this activity he was engaged together with another collaborators of the PTI and of the University in Nottingham in England in the experimental attempts to achieve the Bose-Einstein condensation of the trapped indirect excitons in the double quantum well structures. The efforts of many experimental groups were concentrated many years in this direction, but this topic remains up till now as a desideratum. In the last 12 years Evgenii Moskalenko simultaneously with his activity in PTI took part as an invited researcher in the investigations of the optical properties of the semiconductor nanostructures organized by Professor P.O.Holtz in the Department of Physics and Measurement Technology of the Linkoping University in Sweden. The phenomena of the optical orientation and spin polarization of electrons and nuclei in quantum dots were revealed.

These investigations were influenced by the success of the theory of superconductivity developed originally by Bardin, Cooper and Schriffer,[6] refined later by Bogoliubov[1] as well as by the microscopic theory of superfluidity proposed by Bogoliubov.[1] The specific implementation of these concepts and theorems in the case of 2D magnetoexcitons with direct implication of the plasmon-type excitations side-by-side with the exciton-type branches of the energy spectrum is the main topic of the present review. The coplanar electrons and holes in a strong perpendicular magnetic field at low temperatures form the magnetoexcitons, when the Coulomb interaction between electrons and holes lying on the lowest Landau levels (LLLs) plays the main role. However, when the electrons and holes are spatially separated on the different layers of the double quantum well (DQW) the Coulomb e–h interaction diminishes, and the two-dimensional electron gas (2DEG) on one layer and the two-dimensional hole gas (2DHG) on another layer are formed. Their properties under the conditions of the fractional quantum Hall effect (FQHE) can influence the properties of the 2D magnetoexcitons. To the best of our knowledge these aspects of the magnetoexciton physics were not discussed in literature.

A short review is given on the Bose-Einstein Condensation (BEC) of the quantum Hall excitons (QHExs) arising in the bilayer electron systems under the conditions of the FQHE at one half filling factor for each layer and the total filling factor equal to unity for both layers. This enables us to compare the phenomenon of the BEC of coplanar magnetoexcitons and of QHExs. Such comparison provides better understanding of the underlying physics and allows to verify accuracy of the made approximations. Because the point vortices play an important role in the understanding of the FQHE the corresponding additional information should be included. The possibility to consider the BEC at $T = 0$ as an estimate for the finite temperatures below the Berezinskii–Kosterlitz–Thouless phase transition is suggested.

The article is organized as follow. In Section 2 the Bogoliubov theory of the quasiaverages is overviewed.

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Section 3 is devoted to the Goldstone theorem. The Nambu-Goldstone modes arising under the condition of BEC of the sodium atoms are enumerated in Section 4. The breaking of the local gauge symmetry and the Higgs phenomenon are discussed in Section 5. Section 6 is devoted to the quasi-Nambu-Goldstone modes. In Section 7 the Ginzburg-Landau theory for the FQHE is formulated. The 2D point quantum vortices are described in Section 8. The existence of the statistical gauge vector-potential generated by the vortices is considered in Section 9. The BEC of QHExs and the energy spectrum of elementary excitations under these conditions are discussed in Section 10. Section 11 contains the main results concerning the energy spectrum of the exciton and plasmon branches of the collective elementary excitations of the Bose-Einstein condensed coplanar magnetoexcitons. The spontaneous coherence in 2D excitonic systems is discussed in Section 12. Section 13 is devoted to the Conclusions.

2. BOGOLIUBOV'S THEORY OF QUASIAVERAGES

Bogoliubov[1] has demonstrated his concept of quasiaverages using the ideal Bose-gas model with the Hamiltonian

$$H = \sum_k \left(\frac{\hbar^2 k^2}{2m} - \mu\right) a_k^\dagger a_k \tag{1}$$

here $a_k^\dagger, a_k$ are the Bose operators of creation and annihilation of particles, and μ is their chemical potential.

The occupation numbers of the particles are

$$N_0 = \frac{1}{e^{-\beta\mu} - 1}; \quad N_k = \frac{1}{e^{\beta(\hbar^2 k^2/2m - \mu)} - 1} \tag{2}$$

where $\mu \le 0$ and $\beta = 1/kT$.

In the normal state, the density of particles in the thermodynamic limit at $\mu = 0$ becomes $n = 2.612(mk_B T)^{3/2}/(2\pi\hbar^2)^{3/2}$. At this point, the Bose-Einstein condensation occurs and a finite value of the density of condensed particles appears in the thermodynamic limit

$$n_0 = \lim_{V\to\infty} \frac{N_0}{V}; \quad \mu = -k_B T \ln\left(1 + \frac{1}{N_0}\right) \tag{3}$$

The operators $a_0^\dagger$ and a_0 asymptotically become c-numbers, when their commutator

$$\left[\frac{a_0}{\sqrt{V}}, \frac{a_0^\dagger}{\sqrt{V}}\right] = \frac{1}{V} \tag{4}$$

asymptotically tends to zero and their product is equal to n_0. Then one can write

$$\frac{a_0^\dagger}{\sqrt{V}} \sim \sqrt{n_0} e^{i\alpha}; \quad \frac{a_0}{\sqrt{V}} \sim \sqrt{n_0} e^{-i\alpha} \tag{5}$$

On the other hand, the regular averages of the operators $a_0^\dagger$ and a_0 in the Hamiltonian (1) are exactly equal to zero. It is the consequence of the commutativity of the operator H and the operator of the total particle number N as follows

$$\hat{N} = \sum_k a_k^\dagger a_k; \quad [H, \hat{N}] = 0 \tag{6}$$

As a result, the operators H is invariant with respect to the unitary transformation

$$U = e^{i\hat{N}\phi} \tag{7}$$

with an arbitrary angle ϕ. This invariance is called gradient invariance of the first kind or gauge invariance. When ϕ does not depend on the coordinate x, we have the global gauge invariance and in the case $\phi(x)$ it is called local gauge invariance[2–8] or gauge invariance of the second kind.

The invariance (7) implies $H = U^\dagger H U$; $U^\dagger a_0 U = e^{i\phi} a_0$, which leads to the following average value.

$$\langle a_0\rangle \cong Tr(a_0 e^{-\beta H}) = Tr(a_0 U e^{-\beta} U^\dagger) = Tr(U^\dagger a_0 U e^{-\beta H})$$
$$= e^{i\phi}\langle a_0\rangle; \quad (1 - e^{i\phi})\langle a_0\rangle = 0$$

Because ϕ is an arbitrary angle, there are the selection rules:

$$\langle a_0\rangle = 0; \quad \langle a_0^\dagger\rangle = 0 \tag{8}$$

The regular average (8) can also be obtained from the asymptotical expressions (5) if they are integrated over the angle α. This apparent contradiction can be resolved if Hamiltonian (1) is supplemented by additional term

$$-\nu(a_0^\dagger e^{i\varphi} + a_0 e^{-i\varphi})\sqrt{V}, \quad \nu > 0 \tag{9}$$

where φ is the fixed angle and ν is infinitesimal value.

New Hamiltonian has the form

$$H_{\nu,\phi} = \sum_k \left(\frac{\hbar^2 k^2}{2m} - \mu\right) a_k^\dagger a_k - \nu(a_0^\dagger e^{i\varphi} + a_0 e^{-i\varphi})\sqrt{V} \tag{10}$$

It does not conserve the condensate number. Now the regular average values of the operators $a_0^\dagger$ and a_0 over the Hamiltonian $H_{\nu,\phi}$ differ from zero, i.e., $\langle a_0\rangle_{H\varphi} \neq 0$ and $\langle a_0^\dagger\rangle_{H\varphi} \neq 0$. The definition of the quasiaverages designated by $\langle a_0\rangle$ is the limit of the regular average $\langle a_0\rangle_{H\varphi}$ when ν tends to zero

$$\langle a_0\rangle = \lim_{\nu\to 0}\langle a_0\rangle_{H_{\nu,\varphi}} \tag{11}$$

It is important to emphasize that the limit $\nu \to 0$ must be effectuated after the thermodynamic limit $V \to \infty$, $N_0 \to \infty$. In the thermodynamic limit, μ is also infinitesimal, and it is possible to choose the ratio of two infinitesimal values μ and ν obtaining a finite value

$$-\frac{\nu}{\mu} = \sqrt{n_0} \tag{12}$$

To calculate the regular average $\langle a_0\rangle_{H_{\nu,\varphi}}$ one needs to represent the Hamiltonian (10) $H_{\nu,\phi}$ in a diagonal form with the aid of the canonical transformation over the amplitudes

$$a_0 = -\frac{\nu}{\mu} e^{i\varphi}\sqrt{V} + \alpha_0; \quad a_k = \alpha_k; \quad k \neq 0 \tag{13}$$

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In terms of the new variables the Hamiltonian $H_{\nu,\phi}$ takes the form

$$H_{\nu,\phi} = -\mu\alpha_0^\dagger\alpha_0 + \sum_k\left(\frac{\hbar^2k^2}{2m} - \mu\right)\alpha_{\bar{k}}^\dagger\alpha_{\bar{k}} + \frac{\nu^2 V}{\mu} \tag{14}$$

In the diagonal representation (14), the regular average value $\langle\alpha_0\rangle_{H_{\nu,\varphi}}$ exactly equals to zero, while the value $\langle a_0\rangle_{H_{\nu,\varphi}}$ is equal to the first term on the right-hand side of formulas (13).

As a result, the quasiaverage $\langle a_0\rangle$ is

$$\langle a_0\rangle = \lim_{\nu\to 0}\langle a_0\rangle_{H_{\nu,\varphi}} = \sqrt{N_0}e^{i\varphi} \tag{15}$$

It depends on the fixed angle φ and does not depend on ν. The spontaneous global gauge symmetry breaking is implied when the phase φ of the condensate amplitude in Hamiltonian (10) is fixed.

When the interaction between the particles is taken into account, these differences appear for other amplitudes as well. They give rise to the renormalization of the energy spectrum of the collective elementary excitations. In such a way, the canonical transformation

$$a_k = \sqrt{N_0}\delta_{k,0}e^{i\varphi} + \alpha_k \tag{16}$$

introduced for the first time by Bogoliubov[1] in his theory of superfluidity, has a quantum-statistical foundation within the framework of the quasiaverage concept. At $T = 0$ the quasiaverage $\langle a_0\rangle$ coincides with the average over the quantum-mechanical ground state, which is the coherent macroscopic state.[9]

The phenomena related to the spontaneous breaking of the continuous symmetry play an important role in statistical physics.

Some elements of this concept, such as the coherent macroscopic state with a given fixed phase and the displacement canonical transformation of the field operator describing the Bose-Einstein condensate, were introduced by Bogoliubov in the microscopical theory of superfluidity[1] and were generalized in his theory of quasiaverages[1] noted above.

The brief review of the gauge symmetries, their spontaneous breaking, Goldstone and Higgs effects will be presented below following the Ryder's monograph[7] and Berestetskii's lectures.[8]

3. GOLDSTONE'S THEOREM

Goldstone has considered a simple model of the complex scalar Bose field to demonstrate his main idea. In the classical description the Lagrangian is

$$L = \left(\frac{\partial\phi^*}{\partial x_\mu}\right)\left(\frac{\partial\phi}{\partial x^\mu}\right) - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 \tag{17}$$

The potential energy $V(\phi)$ has the form

$$V(\phi) = m^2\phi^*\phi + \lambda(\phi^*\phi)^2; \quad \lambda > 0 \tag{18}$$

where m^2 is considered as a parameter only, rather than a mass term, λ is the parameter of self-interaction, whereas the denotations x_μ and x^μ mean

$$x^\mu = (ct, \vec{x}); \quad x_\mu = (ct, -\vec{x}) \tag{19}$$

The Lagrangian is invariant under the global gauge transformation

$$\phi = e^{i\wedge}\phi'; \quad L(\phi) = L(\phi'); \quad \Lambda\text{—constant} \tag{20}$$

It has the global gauge symmetry. The ground state is obtained by minimizing the potential as follows.

$$\frac{\partial V(\phi)}{\partial\phi} = m^2\phi^* + 2\lambda\phi^*|\phi|^2 \tag{21}$$

Of interest is the case $m^2 < 0$, when the minima are situated along the ring

$$|\phi|^2 = -\frac{m^2}{2\lambda} = a^2; \quad |\phi| = a; \quad a > 0 \tag{22}$$

The function $V(\phi)$ is shown in Figure 1 being plotted against two real components of the fields ϕ_1 and ϕ_2.

There is a set of degenerate vacua related to each other by rotation. The complex scalar field can be expressed in terms of two scalar real fields, such as $\rho(x)$ and $\theta(x)$, in polar coordinates representation or in the Cartesian decomposition as follows

$$\phi(x) = \rho(x)e^{i\theta(x)} = (\phi_1(x) + i\phi_2(x))\frac{1}{\sqrt{2}} \tag{23}$$

The Bogoliubov-type canonical transformation breaking the global gauge symmetry is

$$\phi(x) = a + \frac{\phi_1'(x) + i\phi_2'(x)}{\sqrt{2}} = (\rho'(x) + a)e^{i\theta'(x)} \tag{24}$$

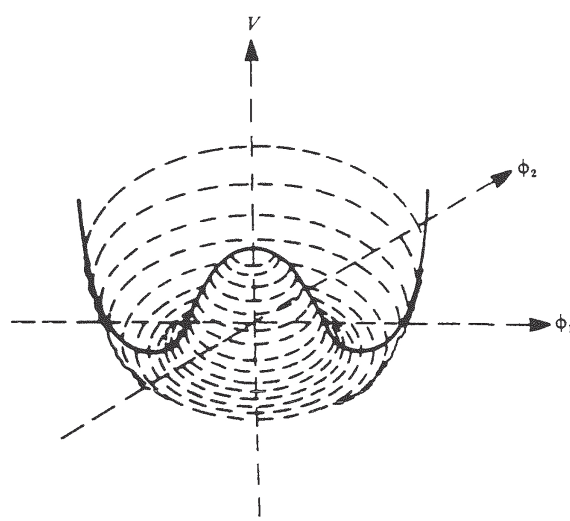

Fig. 1. The potential $V(\phi)$ with the minima at $|\phi| = a$ and a local maximum at $\phi = 0$.

The new particular vacuum state has the average $\langle\phi\rangle_0 = a$ with the particular vanishing vacuum expectation values $\langle\phi_1'\rangle_0 = \langle\phi_2'\rangle_0 = \langle\rho'\rangle_0 = \langle\theta'\rangle_0 = 0$. It means the selection of one vacuum state with infinitesimal phase $\theta' \to 0$. As was pointed in Ref. [7], the physical fields are the excitations above the vacuum. They can be realized by performing perturbations about $|\phi| = a$. Expanding the Lagrangian (17) in series of the infinitesimal perturbations θ', ρ', ϕ_1', ϕ_2' and neglecting by the constant terms, we obtain

$$L = \frac{1}{2}(\partial_\mu\phi_1')(\partial^\mu\phi_1') + \frac{1}{2}(\partial_\mu\phi_2')(\partial^\mu\phi_2') - 2\lambda a^2\phi_1'^2 - \sqrt{2}\lambda\phi_1'(\phi_1'^2+\phi_2'^2) - \frac{\lambda}{4}(\phi_1'^2+\phi_2'^2)^2 \quad (25)$$

or in polar description

$$L = (\partial_\mu\rho')(\partial^\mu\rho') + (\rho'+a)^2(\partial_\mu\theta')(\partial^\mu\theta') - [\lambda\rho'^4 + 4a\lambda\rho'^3 + 4\lambda a^2\rho'^2 - \lambda a^4] \quad (26)$$

Neglecting by the cubic and quartic terms, we will see that there are the quadratic terms only of the type $4\lambda a^2\rho'^2$ and $2\lambda a^2\phi_1'^2$, but there is no quadratic terms proportional to θ'^2 and $\phi_2'^2$. For real physical problems, for example, for the field theory, the field components ϕ_1' and ρ' represent massive particles and dispersion laws with energy gap, whereas the field components ϕ_2' and θ' represent the massless particles and gapless energy spectrum.

The main Goldstone results can be formulated as follows.

$$m_{\rho'}^2 = 4\lambda a^2; \quad m_{\phi_1'}^2 = 2\lambda a^2$$
$$m_{\theta'}^2 = 0; \quad m_{\phi_2'}^2 = 0 \quad (27)$$

The spontaneous breaking of the global gauge symmetry takes place due to the influence of the quantum fluctuations. They transform the initial field ϕ with two massive real components ϕ_1 and ϕ_2, and a degenerate ground state with the minima forming a ring into another field with one massive and other massless components, the ground state of which has a well defined phase without initial symmetry.

The elementary excitations above the new ground state changing the value $\langle\rho\rangle = a$ are massive. It costs energy to displace ρ' against the restoring forces of the potential $V(\rho)$. But there are no restoring forces corresponding to displacements along the circular valley $|\phi| = a$ formed by initial degenerate vacua.

Hence, for angular excitations θ' of wavelength, λ we have $\omega \sim \lambda^{-1} \to 0$ as $\lambda \to \infty$. The dispersion law is $\omega \sim ck$ and the particles are massless.[7] The θ' particles are known as the Goldstone bosons. This phenomenon is general and takes place in any order of perturbation theory. The spontaneous breaking of a continuous symmetry not only of the type as a global gauge symmetry but also of the type of rotational symmetry entails the existence of massless particles referred to as Goldstone particles or Nambu-Golstone gapless modes. This statement is known as Goldstone theorem. It establishes that there exists a gapless excitation mode when a continuous symmetry is spontaneously broken. The angular excitations θ' are analogous to the spin waves. The latter represent a slow spatial variation of the direction of magnetization without changing of its absolute value. Since the forces in a ferromagnetic are of short range, it requires a very little energy to excite this ground state. So, the frequency of the spin waves has the dispersion law $\omega = ck$. As was mentioned by Ryder,[7] this argument breaks down if there are long-range forces like, for example, the $1/r$ Coulomb force. In this case, we deal with the maxwellian gauge field with local depending on x gauge symmetry instead of global gauge symmetry considered above.

After the specific application of the above statement will be demonstrated following Refs. [10–16], where the spinor Bose-Einstein condensates were discussed, we will consider the case of Goldstone field ϕ and of the maxwellian field with local gauge symmetry.

4. BOGOLIUBOV'S EXCITATIONS AND THE NAMBU-GOLDSTONE MODES

The above formulated theorems can be illustrated using the specific example of the Bose-Einstein condensed sodium atoms ^{23}Na in an optical-dipole trap following the investigations of Murata, Saito and Ueda[10] on the one side and of Uchino, Kobayashi and Ueda[11] on the other side. There are numerous publications on this subject among which should be mentioned.[12–17] The sodium atom ^{23}Na has spin $f = 1$ of the hyperfine interaction and obey the Bose statistics. Resultant spin of the interacting bosons with $f = 1$ is F which takes the values $F = 0, 1, 2$. The contact hard-core interaction constant $g_F = 4\pi\hbar^2 a_F/M$ is characterized by s-wave scattering length a_F, which is not zero for $F = 0$ when two atomic spins form a singlet, and for $F = 2$, when they form a quintuplet. The constant g_0 and g_2 enter the combinations $c_0 = (g_0 + 2g_2)/3$ and $c_1 = (g_2 - g_0)/3$ which determine the Hamiltonian. The description of the atomic Bose gas in an optical-dipole trap is possible in the plane-wave representation due to the homogeneity and the translational symmetry of the system. It means that the components of the Bose field operator $\psi_m(\vec{r})$ can be represented in the form:

$$\psi_m(\vec{r}) = \frac{1}{\sqrt{V}}\sum_k a_{km} e^{i\vec{k}\vec{r}} \quad (28)$$

where V is the volume of the system, and a_{km} is the annihilation operator with the wave vector $\vec{k}$ and the magnetic quantum number m, which in the case $f = 1$ takes three values $1, 0, -1$. The spinor Bose-Einstein condensates were realized experimentally by the MIT group[12] for different spin combinations using the sodium atoms ^{23}Na

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in a hyperfine spin states $|f=1, m_f=-1\rangle$ in a magnetic trap and then transforming them to the optical-dipole trap formed by the single infrared laser. The Bose-Einstein condensates were found to be long-lived. Some arguments concerning the metastable long-lived states were formulated. The states may appear if the energy barriers exist, which prevent the system from direct evolving toward its ground states. If the thermal energy needed to overcome these barriers is not available, the metastable state may be long-lived and these events are commonly encountered. Even the Bose-Einstein condensates in the dilute atomic gases can also be formed due to the metastability. Moreover, in the gases with attractive interactions the Bose-Einstein condensates may be metastable against the collapse just due to the energy barriers.[12] Bellow we will discuss the Bogoliubov-type collective elementary excitations arising over the metastable long-lived ground states of the spinor-type Bose-Einstein condensates (BEC-tes) following Refs. [10, 11], so as to demonstrate the formation of the Nambu-Goldstone modes.

The Hamiltonian considered in Ref. [10] is given by formulas (3) and (4), and has the form

$$H=\sum_{\vec{k},m}(\varepsilon_{\vec{k}}-pm+qm^2)a^{\dagger}_{\vec{k}m}a_{\vec{k}m} + \frac{c_0}{2V}\sum_{\vec{k}}:\hat{\rho}^{\dagger}_{\vec{k}}\hat{\rho}_{\vec{k}}:+\frac{c_1}{2V}\sum_{\vec{k}}:\hat{\vec{f}}^{\dagger}_{\vec{k}}\hat{\vec{f}}_{\vec{k}}: \tag{29}$$

Here the following designations were used

$$\varepsilon_k=\tfrac{\hbar^2k^2}{2M};\quad c_0=(g_0+2g_2)/3;\quad c_1=(g_2-g_0)/3$$
$$\hat{\rho}_{\vec{k}}=\textstyle\sum_{\vec{q},m}a^{\dagger}_{\vec{q},m}a_{\vec{q}+\vec{k},m};\quad \hat{\vec{f}}=(\hat{f}^x,\hat{f}^y,\hat{f}^z)$$
$$\hat{\vec{f}}_{\vec{k}}=\textstyle\sum_{q,m,n}\hat{\vec{f}}_{mn}a^{\dagger}_{q,m}a_{q+k,m}$$
$$\hat{f}^x=\begin{vmatrix}0&1&0\\1&0&1\\0&1&0\end{vmatrix}\frac{1}{\sqrt{2}};\quad \hat{f}^y=\begin{vmatrix}0&-1&0\\1&0&-1\\0&1&0\end{vmatrix}\frac{i}{\sqrt{2}} \tag{30}$$
$$\hat{f}^x=\begin{vmatrix}1&0&0\\0&0&0\\0&0&-1\end{vmatrix}$$

The repeated indexes mean summation over $1, 0, -1$. The symbol :: denotes the normal ordering of the operators. The coefficient p is the sum of the linear Zeeman energy and of the Lagrangian multiplier, which is introduced to set the total magnetization in the z direction to a prescribed value. This magnetization is conserved due to the axisymmetry of the system in a magnetic field. q is the quadratic Zeeman effect energy, which is positive in the case of spin $f=1$ for ^{23}Na and ^{87}Rb atoms. The spin-spin interaction is of ferromagnetic-type with $c_1<0$ for $f=1$ ^{87}Rb atoms and is antiferromagnetic-type with $c_1>0$ for $f=1$ ^{23}Na atoms.[10] Taking into account that in many experimental situations the linear Zeeman effect can be ignored and the quadratic Zeeman effect term q can be manipulated experimentally, in Ref. [11] both cases of positive and negative q at $p=0$ were investigated for spin 1 and spin 2 Bose-Einstein condensates (BECs). We restrict ourselves to review some spinor phases with spin 1 discussed in Ref. [11] so as to demonstrate the relations between the Nambu-Goldstone (NG) modes of the Bogoliubov energy spectra and the spontaneous breaking of the continuous symmetries. The description of the excitations is presented in Refs. [10, 11] in the number-conserving variant of the Bogoliubov theory.[1] There is no need to introduce the chemical potential as a Lagrangian multiplier in order to adjust the particle number to a prescribed value. The BEC takes place on a superposition state involving the single-particle states with wave vector $\vec{k}=0$ and different magnetic quantum numbers

$$|\xi\rangle=\sum_m \xi_m a^{\dagger}_{0,m}|vac\rangle;\quad \sum_m|\xi_m|^2=1 \tag{31}$$

The order parameter has a vector form and consists of three components: $\vec{\xi}=(\xi_1,\xi_0,\xi_{-1})$. The vacuum state $|vac\rangle$ means the absence of the atoms. The ground state wave function of the BEC-ed atoms is given by the formula (8) of Ref. [11]

$$|\psi_g\rangle=\frac{1}{\sqrt{N!}}\left(\sum_{m=-f}^{f}\xi_m a^{\dagger}_{0,m}\right)^N|vac\rangle \tag{32}$$

In the mean-field approximation the operators $a^{\dagger}_{0,m}$, $a_{0,m}$ are replaced by the c-numbers $\xi_m\sqrt{N_0}$, where N_0 is the number of the condensed atoms. After this substitution, the initial Hamiltonian loses its global gauge symmetry and does not commute with the operator $\hat{N}$. The order parameters ξ_m are chosen so that to minimize the expectation value of the new Hamiltonian and its ground state and to satisfy the normalization condition $\sum_m|\xi_m|^2=1$. To keep the order parameter of each phase unchanged it is necessary to specify the combination of the gauge transformation and spin rotations.[11] This program was carried out in Refs. [18–21].

The initial Hamiltonian (29) in the absence of the external magnetic field has the symmetry $U(1)\times SO(3)$ representing the global gauge symmetry $U(1)$ and the spin-rotation symmetry $SO(3)$. The generators of these symmetries are referred to as symmetry generators and have the form

$$\hat{N}=\int d\vec{x}\hat{\psi}^{\dagger}_m(x)\hat{\psi}_m(x)=\sum_{\vec{k},m}a^{\dagger}_{\vec{k},m}a_{\vec{k},m}$$
$$\hat{F}^j=\int d\vec{x}\hat{\psi}_m(x)f^j_{mn}\hat{\psi}_n(x);\quad j=x,y,z \tag{33}$$

Unlike the $SO(3)$ symmetry group with three generators $\hat{F}^x$, $\hat{F}^y$ and $\hat{F}^z$, the $SO(2)$ symmetry group has only one generator $\hat{F}^z$ which describes the spin rotation around the z axis and looks as follows:

$$\hat{F}^z = \sum_{\vec{k},m} m a^{\dagger}_{\vec{k},m} a_{\vec{k},m} \tag{34}$$

In the presence of an external magnetic field, the symmetry of the Hamiltonian is $U(1)\times SO(2)$. The breaking of the continuous symmetry means the breaking of their generators. The number of the broken generators (BG) is denoted as N_{BG}. There are 4 generators in the case of $U(1)\times SO(3)$ symmetry and two in the case of $U(1)\times SO(2)$ symmetry.

The phase transition of the spinor Bose gas from the normal state to the Bose-Einstein condensed state was introduced mathematically into Hamiltonian (29) using the Bogoliubov displacement canonical transformation, when the single-particle creation and annihilation operators with the given wave vector $\vec{k}$, for example, $\vec{k}=0$, were substituted by the macroscopically c-numbers describing the condensate formation. The different superpositions of the single-particle states determine the structure of the finally established spinor phases.[11] Nielsen and Chadha[17] formulated the theorem which establishes the relation between the number of the Nambu-Goldstone modes, which must be present between the amount of the collective elementary excitations, which appear over the ground state of the system if it is formed as a result of the spontaneous breaking of the N_{BG} continuous symmetries. The number of NG modes of the first type with linear (odd) dispersion law in the limit of long wavelengths denoted as N_I being accounted once, and the number N_{II} of the NG modes of the second type with quadratic (even) dispersion law at small wave vectors, being accounted twice give rise to the expression $N_I + 2N_{II}$, which is equal to or greater than the number N_{BG} of the broken symmetry generators. The theorem[17] says

$$N_I + 2N_{II} \geq N_{BG} \tag{35}$$

The theorem has been verified in Ref. [11] for multiple examples of the spin 1 and spin 2 Bose-Einstein condensate phases. In the case of spin 2 nematic phases, the special Bogoliubov modes that have linear dispersion relation but do not belong to the NG modes were revealed. The Bogoliubov theory of the spin 1 and spin 2 Bose-Einstein condensates (BECs) in the presence of the quadratic Zeeman effect was developed by Uchino, Kobayashi and Ueda[11] taking into account the Lee, Huang, Yang (LHY) corrections to the ground state energy, pressure, sound velocity and quantum depletion of the condensate. Many phases that can be realized experimentally were discussed to examine their stability against the quantum fluctuations and the quadratic Zeeman effect. The relations between the numbers of the NG modes and of the broken symmetry generators were verified. A brief review of the results concerning the spin 1 phases of Ref. [11] is presented below so as to demonstrate, using these examples, the relations between the Bogoliubov excitations and the Nambu-Goldstone modes.

The first example is the ferromagnetic phase with $c_1 < 0$, $q < 0$ and the vector order parameter

$$\vec{\xi}^F = (1, 0, 0) \tag{36}$$

The modes with $m=0$ and $m=-1$ are already diagonalized, whereas the mode $m=1$ is diagonalized by the standard Bogoliubov transformation. The Bogoliubov spectrum is given by formulas (33) and (34) of Ref. [11]

$$\begin{gathered} E_{\vec{k},1} = \sqrt{\varepsilon_{\vec{k}}(\varepsilon_{\vec{k}} + 2\eta(c_0 + c_1))}; \quad E_{\vec{k},0} = \varepsilon_{\vec{k}} - q \\ E_{\vec{k},-1} = \varepsilon_{\vec{k}} - 2c_1 n \end{gathered} \tag{37}$$

The $E_{\vec{k},1}$ mode is massless. In the absence of a magnetic field, when $q=0$, the mode $m=0$ is also massless with the quadratic dispersion law. The initial symmetry of the Hamiltonian before the phase transition is $U(1)\times SO(3)$, whereas the final, remaining symmetry after the process of BEC is the symmetry of the ferromagnetic i.e., $SO(2)$. From the four initial symmetry generators $\hat{N}$, $\hat{F}^x$, $\hat{F}^y$ and $\hat{F}^z$ remains only the generator $\hat{F}^z$ of the $SO(2)$ symmetry. The generators $\hat{F}^x$ and $\hat{F}^y$ were broken by the ferromagnet phase, whereas the gauge symmetry operator $\hat{N}$ was broken by the Bogoliubov displacement transformation. The number of the broken generators $\hat{N}$, $\hat{F}^x$, $\hat{F}^y$ is three, i.e., $N_{BG}=3$. In this case $N_I=1$, $N_{II}=1$ and $N_I+2N_{II}=3$, being equal to $N_{BG}=3$. The equality $N_I+2N_{II}=N_{BG}$ takes place. In the presence of an external magnetic field, with $q\neq 0$, the initial symmetry before the phase transition is $U(1)\times SO(2)$ with two generators $\hat{N}$ and $\hat{F}^z$, whereas after the BEC and the ferromagnetic phase formation the remained symmetry is $SO(2)$. Only one symmetry generator $\hat{N}$ was broken. It means $N_{BG}=1$, $N_I=1$ and $N_{II}=0$. The equality $N_I+2N_{II}=N_{BG}$ also takes place.

The condition $(c_0+c_1)>0$ to be hold is required for $m=1$ the Bogoliubov mode to be stable. It ensures the mechanical stability of the mean-field ground state. Otherwise, the compressibility would not be positive definite and the system would become unstable against collapse. In the case $q>0$, $c_1>0$ and $(c_0+c_1)<0$ the state would undergo the Landau instability for the $m=0$ and $m=-1$ modes with quadratic spectra and the dynamical instability for the $m=1$ mode with a linear spectrum (36) of Ref. [11].

There are two polar phases. One with the parameters

$$\vec{\xi}^P = (0, 1, 0); \quad q > 0; \quad q + 2nc_1 > 0 \tag{38}$$

and the other with the parameters

$$\vec{\xi}^P = \frac{1}{\sqrt{2}}(1, 0, 1); \quad q < 0; \quad c_1 > 0 \tag{39}$$

These two polar phases have two spinor configurations which are degenerate at $q=0$ and connect other by

$U(1)\times SO(3)$ transformation. However, for nonzero q the degeneracy is lifted and they should be considered as different phases. This is because the phase P has a remaining symmetry $SO(2)$, whereas the phase P' is not invariant under any continuous transformation. The number of NG modes is different in each phase and the low-energy behavior is also different. Following formulas (40)–(42) of Ref. [11] the density fluctuation operator a_{kd} and the spin fluctuation operators a_{k,f_x} and a_{k,f_y} were introduced

$$a_{kd}=a_{k,0};\quad a_{k,f_x}=\frac{1}{\sqrt{2}}(a_{k,1}+a_{k,-1})$$
$$a_{k,f_y}=\frac{i}{\sqrt{2}}(a_{k,1}-a_{k,-1}) \tag{40}$$

Their Bogoliubov energy spectra are

$$E_{\vec{k},d}=\sqrt{\varepsilon_{\vec{k}}(\varepsilon_{\vec{k}}+2c_0)}$$
$$E_{\vec{k},f_j}=\sqrt{(\varepsilon_{\vec{k}}+q)(\varepsilon_{\vec{k}}+q+2nc_1)} \tag{41}$$

In the presence of an external magnetic field, the initial symmetry is $U(1)\times SO(2)$, whereas after the BEC and the formation of the phase P with $q\neq 0$ the remaining symmetry is also $SO(2)$. Only the symmetry $U(1)$ and its generator $\hat{N}$ were broken during the phase transition. It means we have in this case $N_{BG}=1$, $N_I=1$ and $N_{II}=0$. The equality $N_I+2N_{II}=N_{BG}$ holds. Density mode is massless because the $U(1)$ gauge symmetry is spontaneously broken in the mean-field ground state, while the transverse magnetization modes f_x and f_y are massive for non zero q, since the rotational degeneracies about the x and y axes do not exist being lifted by the external magnetic field. In the limit of infinitesimal $q\to 0$ nevertheless nonzero, the transverse magnetization modes f_x and f_y become massless. It occurs because before the BEC in the absence of an external magnetic field the symmetry of the spinor Bose gas is $U(1)\times SO(3)$, whereas after the phase transition it can be considered as a remaining symmetry $SO(2)$. The generators $\hat{N}$, $\hat{F}^x$, $\hat{F}^y$ were broken, whereas the generator $\hat{F}^z$ remained. In this case we have $N_{BG}=3$, $N_I=3$ and $N_{II}=0$ the equality looks as $3=3$.

In the polar phase P' with the parameters (39) the density and spin fluctuation operators were introduced by formulas (57)–(59) of Ref. [11]

$$a_{kd}=\frac{1}{\sqrt{2}}(a_{k,1}+a_{k,-1});\quad a_{k,f_x}=a_{k,0}$$
$$a_{k,f_y}=\frac{i}{\sqrt{2}}(a_{k,1}-a_{k,-1}) \tag{42}$$

with the Bogoliubov energy spectra described by formulas (65)–(67):[11]

$$E_{\vec{k},d}=\sqrt{\varepsilon_{\vec{k}}(\varepsilon_{\vec{k}}+2nc_0)}$$
$$E_{\vec{k},f_x}=\sqrt{(\varepsilon_{\vec{k}}-q)(\varepsilon_{\vec{k}}-q+2nc_1)} \tag{43}$$
$$E_{\vec{k},f_z}=\sqrt{\varepsilon_{\vec{k}}(\varepsilon_{\vec{k}}+2nc_1)}$$

At $q<0$ in contrast to the case $q>0$ one of the spin fluctuation mode $E_{\vec{k},f_z}$ becomes massless. The initial symmetry of the system is $U(1)\times SO(2)$. It has the symmetry generators $\hat{N}$ and $\hat{F}^z$. They are completely broken during the phase transition. After the phase transition and the P' phase formation there are not any symmetry generators. The number of the broken generator is 2 ($N_{BG}=2$), whereas the numbers N_I and N_{II} are 2 and 0, respectively. As in the previous cases, the equality occurs in the Nielsen and Chadha rule. For the Bogoliubov spectra to be real the condition $q<0$, $c_0>0$ and $c_1>0$ must be satisfied, otherwise, the state $\vec{\xi}^{P'}$ will be dynamically unstable.

Side by side with the spinor-type three-dimensional (3D) atomic Bose-Einstein condensates in the optical traps, we will discuss also the case of the Bose-Einstein condensation of the two-dimensional (2D) magnetoexcitons in semiconductors.[22–25] The collective elementary excitations under these conditions were investigated in Refs. [26–31] and will be described in Section 11. As was shown above, the spontaneous symmetry breaking yields Nambu-Goldstone modes, which play a crucial role in determining low-energy behavior of various systems.[5, 32–38] Side by side with the global gauge symmetry the local symmetry does exist.

5. SPONTANEOUS BREAKING OF THE LOCAL GAUGE SYMMETRY AND THE HIGGS PHENOMENON

The interaction of the electrons with the electromagnetic field can be described introducing into the Lagrangian the kinetic momentum operators instead of canonical ones what is equivalent to introduce the covariant derivatives D instead of the differential ones ∂. They are determined in Ref. [8] as

$$\underline{x}=(ct,\vec{x}),\quad \underline{\partial}=\left(\frac{1}{c}\frac{\partial}{\partial t},\vec{\nabla}\right)$$
$$\underline{D}=\underline{\partial}-\frac{ie}{\hbar c}\underline{A};\quad \underline{A}=(\varphi,\vec{A}) \tag{44}$$

where φ and $\vec{A}$ are the scalar and vector potentials of the electromagnetic field (EMF). Below we will use also the denotations of Ref. [7]

$$x^\mu=(ct,\vec{r});\quad x_\mu=(ct,-\vec{r});\quad A_\mu=(\varphi,-\vec{A})$$
$$A^\mu=(\varphi,\vec{A});\quad \partial_\mu=\frac{\partial}{\partial x^\mu}=\left(\frac{1}{c}\frac{\partial}{\partial t},\vec{\nabla}\right)$$
$$\partial^\mu=\frac{\partial}{\partial x_\mu}\left(\frac{1}{c}\frac{\partial}{\partial t},-\vec{\nabla}\right);\quad \partial_\mu\partial^\mu=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta \tag{45}$$
$$p^\mu=\left(\frac{E}{c},\vec{p}\right);\quad p_\mu=\left(\frac{E}{c},-\vec{p}\right)$$

The Lagrangian of the free EMF has the form[7]

$$\mathscr{L}_{EMF} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (46)$$

being expressed through the antisymmetric tensors $F_{\mu\nu}$ and $F^{\mu\nu}$. They are determined as four-dimensional curls of A_μ and A^μ.

$$F_{\mu\nu} = -F_{\nu\mu} = \partial_\mu A_\nu - \partial_\nu A_\mu; \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad (47)$$

The full Lagrangian of the electrons and EMF reads[7]

$$L = \left[\left(\partial_\mu + \frac{ie}{\hbar c}A_\mu\right)\phi\right]\left[\left(\partial - \frac{ie}{\hbar c}A^\mu\right)\phi^*\right] - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (48)$$

As before m^2 is a parameter so that in the case $m^2 < 0$ and in the absence of the EMF vacuum values are determined by the formula (22).

The invariance of the Lagrangian (48) under the transformation[8]

$$\phi'(\underline{x}) = \phi(\underline{x})e^{i\theta(\underline{x})} \quad (49)$$

in the presence of the EMF can be achieved only under the concomitant transformation of its potential in the form[8]

$$\underline{A}'(x) = A(\underline{x}) + \frac{\hbar c}{e}\partial\theta(\underline{x}) \quad (50)$$

Indeed in this case the Lagrangian (48) remains invariant [7] as follows

$$\begin{gathered}\left[\left(\partial_\mu + \frac{ie}{\hbar c}A_\mu\right)\phi\right]\left[\left(\partial^\mu - \frac{ie}{\hbar c}A^\mu\right)\phi^*\right] \\ = \left[\left(\partial_\mu + \frac{ie}{\hbar c}A'_\mu\right)\phi'\right]\left[\left(\partial^\mu - \frac{ie}{\hbar c}A'^\mu\right)\phi'^*\right] \\ F'_{\mu\nu} = F_{\mu\nu}; \quad F'^{\mu\nu} = F^{\mu\nu}\end{gathered} \quad (51)$$

Introducing the gauge transformation of the field function (24) and expanding the Lagrangian in power series on the small physical fields ϕ'_1 and ϕ'_2 we obtain the constant, quadratic, cubic and quartic terms. The quadratic part looks as[7]

$$\mathscr{L}_2 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + e^2a^2A_\mu A^\mu + \frac{1}{2}(\partial_\mu\phi'_1)^2 + \frac{1}{2}(\partial_\mu\phi'_2)^2 - 2\lambda a^2\phi'^2_1 + \sqrt{2}eaA^\mu\partial_\mu\phi'_2 \quad (52)$$

The second term is proportional to $A_\mu A^\mu$. It indicates that the photon becomes massive. The scalar field ϕ'_1 is also a massive one. The field ϕ'_2 takes part in the mixed term $A^\mu\partial_\mu\phi'_2$ and can be eliminated by the supplementary gauge transformation (50). Following Ref. [7] the Lagrangian (52) can be presented in the form

$$\mathscr{L}_2 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + e^2a^2A_\mu A^\mu + \frac{1}{2}(\partial_\mu\phi'_1)^2 - 2\lambda a^2\phi'^2_1 \quad (53)$$

It contains two fields only: the photon with longitudinal component and spin 1 and field ϕ'_1 with spin 0. They are both massive. The field ϕ'_2, which in the case of spontaneous breaking of the global gauge symmetry became massless forming a Goldstone boson, in this case disappeared. The photon became massive. This phenomenon is called the Higgs phenomenon.[7]

One possible illustration of the described above effect will be considered below following the paper by Halperin, Lee and Read.[39] They considered the two-dimensional (2D) system of spinless electrons under the conditions of the quantum Hall effect. Then the Hamiltonian $\hat{H} = \hat{K} + \hat{V}$ consists of the kinetic energy operator $\hat{K}$

$$\hat{K} = \frac{1}{2m_e}\int d^2\vec{r}\hat{\psi}^\dagger_e(\vec{r})\left[-i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}(\vec{r})\right]^2\hat{\psi}_e(\vec{r}) \quad (54)$$

with 2D electrons with the mass m_e and the charge $-e$ situated in a uniform external perpendicular magnetic field B with the vector-potential $\vec{A}(\vec{r})$. The potential energy operator $\hat{V}$ depends on the Coulomb interaction between the electrons. The creation and annihilation operators $\hat{\psi}^\dagger_e(\vec{r})$, $\hat{\psi}_e(\vec{r})$ obey to the Fermi statistics as was the case of Ref. [39], but we will consider following the Ref. [40] a more general case including also the Bose statistics

$$[\hat{\psi}_e(\vec{r})\hat{\psi}^\dagger_e(\vec{r}') \pm \hat{\psi}^\dagger_e(\vec{r}')\hat{\psi}_e(\vec{r})] = \delta^2(\vec{r}-\vec{r}') \quad (55)$$

The signs $\pm$ correspond to the Fermi and Bose statistics. In Ref. [39] the new "quasiparticle" operators $\hat{\psi}^\dagger(\vec{r})$, $\hat{\psi}(\vec{r})$ were introduced by the relations

$$\hat{\psi}^\dagger(\vec{r}) = \hat{\psi}^\dagger_e(\vec{r})e^{-im\hat{\omega}(\vec{r})}; \quad \hat{\psi}(\vec{r}) = e^{im\hat{\omega}(\vec{r})}\hat{\psi}_e(\vec{r}) \quad (56)$$

with an integer number m and with the phase operator

$$\hat{\omega}(\vec{r}) = \int d^2\vec{r}'\theta(\vec{r}-\vec{r}')\hat{\rho}(\vec{r}') \quad (57)$$

It depends on the angle $\theta(\vec{r}-\vec{r}')$ between the vector $\vec{r}-\vec{r}'$ and the in-plane axis x being determined by the formula

$$\theta(\vec{r}-\vec{r}') = \arctan\frac{y-y'}{x-x'} \quad (58)$$

and by the density operator $\hat{\rho}(\vec{r}')$

$$\hat{\rho}(\vec{r}') = \hat{\psi}^\dagger_e(\vec{r}')\hat{\psi}_e(\vec{r}) = \hat{\psi}^\dagger(\vec{r})\hat{\psi}(\vec{r}') \quad (59)$$

These operators have the properties

$$\begin{gathered}\hat{\omega}(\vec{r}) = \hat{\omega}^\dagger(\vec{r}); \quad [\hat{\omega}(\vec{r}), \hat{\omega}(\vec{r}')] = 0 \\ [\hat{\psi}_e(\vec{r}), \hat{\rho}(\vec{r}')] = \hat{\psi}_e(r)\delta^2(\vec{r}-\vec{r}') \\ [\hat{\psi}_e(\vec{r}), \hat{\omega}(\vec{r}')] = \hat{\psi}_e(\vec{r})\theta(\vec{r}'-\vec{r}) \\ \hat{\psi}_e(\vec{r})\hat{\omega}^n(\vec{r}') = (\hat{\omega}(\vec{r}') + \theta(\vec{r}'-\vec{r}))^n\hat{\psi}_e(\vec{r}) \\ \hat{\psi}_e(\vec{r})e^{im\hat{\omega}(r')} = e^{im\theta(\vec{r}'-\vec{r})}e^{im\hat{\omega}(r')}\hat{\psi}_e(\vec{r}) \\ \hat{\psi}^\dagger_e(\vec{r})e^{-im\hat{\omega}(r')} = e^{im\theta(\vec{r}'-\vec{r})}e^{-im\hat{\omega}(r')}\hat{\psi}^\dagger_e(\vec{r})\end{gathered} \quad (60)$$

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It will be shown below that m is the number of point vortices attached to each bare initial particle forming together with it a composite particle (CP). The statistics of the CPs depends on the statistics of the initial particles and on the number m of the attached vortices. Finally, we will calculate the commutators of the operators $\hat{\psi}^\dagger(\vec{r})$, $\hat{\psi}(\vec{r})$ with the requirement that it will be $\delta^2(\vec{r}-\vec{r}')$ as follows:

$$[\hat{\psi}(\vec{r}),\hat{\psi}^\dagger(\vec{r}')]_\pm = e^{-im\theta(0)+im\theta(\vec{r}-\vec{r})} \times[\hat{\psi}_e(\vec{r})\hat{\psi}_e^\dagger(\vec{r}')\pm e^{im\pi}\hat{\psi}_e^\dagger(\vec{r}')\hat{\psi}_e(\vec{r})] \times e^{-im(\hat{\omega}(\vec{r}')-\hat{\omega}(\vec{r}))} = \delta^2(\vec{r}-\vec{r}') \qquad (61)$$

Here we have taken into account the relation $\theta(\vec{r}'-\vec{r})-\theta(\vec{r}-\vec{r}')=\pi$ for $\vec{r}\neq\vec{r}'$. One can observe that the CPs represented by the operators $\hat{\psi}^\dagger(\vec{r})$, $\hat{\psi}(\vec{r})$ are composite fermions (CFs) if the bare initial particles are fermions and the number of vortices m is even as well as in the case when the initial particles are bosons and the number of vortices m is odd. In the same way the CPs are composite bosons (CBs) if the initial particles are fermions and the number of vortices is odd, or if the initial particles are bosons and the number of vortices m is even.

The kinetic energy operator $\hat{K}$ in terms of the operators $\psi^\dagger(r)$ and $\psi(r)$ is

$$\hat{K} = \frac{\hbar^2}{2m_e}\int d^2\vec{r}\psi^\dagger(\vec{r})e^{im\hat{\omega}(\vec{r})}\left[-i\vec{\nabla}+\frac{e}{\hbar c}\vec{A}(\vec{r})\right]^2 \times e^{-im\hat{\omega}(\vec{r})}\psi(\vec{r}) \qquad (62)$$

It can be transformed taking into account that

$$\left(-i\vec{\nabla}+\frac{e}{\hbar c}\vec{A}(\vec{r})\right)e^{-im\hat{\omega}(\vec{r})}\psi(\vec{r}) = e^{-im\hat{\omega}(\vec{r})}\left(-i\vec{\nabla}+\frac{e}{\hbar c}\vec{A}(\vec{r})-m\vec{\nabla}\hat{\omega}(\vec{r})\right)\psi(\vec{r}) \qquad (63)$$

what leads to the formula

$$\hat{K} = \frac{\hbar^2}{2m_e}\int d^2\vec{r}\psi^\dagger(\vec{r}) \times\left[-i\vec{\nabla}+\frac{e}{\hbar c}\vec{A}(\vec{r})-m\vec{\nabla}\hat{\omega}(\vec{r})\right]^2\psi(\vec{r}) \qquad (64)$$

It contains a supplementary vector potential $\hat{\vec{a}}(\vec{r})$ named as statistical Chern-Simons gauge potential[41] determined as

$$\hat{\vec{a}}(\vec{r}) = -\frac{m\hbar c}{e}\vec{\nabla}\hat{\omega}(\vec{r}) = -\frac{m\hbar c}{e}\int d^2\vec{r}'\vec{\nabla}_r\theta(\vec{r}-\vec{r}')\hat{\rho}(\vec{r}') \qquad (65)$$

Its calculation needs a special precaution as was pointed by Jackiw and Pi[41] because $\theta(\vec{r}-\vec{r}')$ is a multivalued function. They cautioned against the moving of $\vec{\nabla}$ with respect to $\vec{r}$ out of the integral $\int d^2\vec{r}'\theta(\vec{r}-\vec{r}')\hat{\rho}(\vec{r}')$, because in general it is not correct. The integration cannot be interchanged with the differentiation. The reason for this is that the function $\theta(\vec{r}-\vec{r}')$ is multivalued and the integration of $\theta(\vec{r}-\vec{r}')$ over the two-dimensional $\vec{r}'$ plane requires specifying the cut in the space $\vec{r}'$, which begins at the point $\vec{r}$. The range of the $\vec{r}'$ integration depends on $\vec{r}$ and moving the $\vec{r}$ derivative outside of the $\vec{r}'$ integral gives an additional contribution. To avoid these complications the derivative $\vec{\nabla}\theta(\vec{r}-\vec{r}')$ is introduced into the integrand for the very beginning in the form

$$\vec{\nabla}\theta(\vec{r}-\vec{r}') = -curl\ln|\vec{r}-\vec{r}'| \qquad (66)$$

As it was shown in Ref. [41] the curl of a scalar S in the 2D space is a vector and the curl of the vector $\vec{a}$ is a scalar as follows

$$(curl S)^i = \in^{ij}\partial_j S;\quad curl\vec{a} = \in^{ij}\partial_i a_j;\quad i,j=1,2$$
$$\in^{12} = -\in^{21};\quad \in^{11}=\in^{22}=0 \qquad (67)$$

The statistical gauge vector potential $\hat{\vec{a}}(\vec{r})$ can be transcibed

$$\hat{\vec{a}}(\vec{r}) = \frac{m\hbar c}{e}\int d^2\vec{r}' curl\ln|\vec{r}-\vec{r}'|\hat{\rho}(\vec{r}') \qquad (68)$$

what leads to the statistical gauge magnetic field $\hat{b}(\vec{r})$

$$\hat{b}(\vec{r}) = curl\hat{\vec{a}}(\vec{r}) = \frac{m\hbar c}{e}\int d^2\vec{r}' \in^{ij}\in^{jk}\partial_i\partial_k\ln|\vec{r}-\vec{r}'|\hat{\rho}(\vec{r}') = -\frac{m\hbar c}{e}\int d^2\vec{r}'\Delta_r\ln|\vec{r}-\vec{r}'|\hat{\rho}(\vec{r}') \qquad (69)$$

Taking into account the equality

$$\Delta\ln\vec{r} = 2\pi\delta^2(\vec{r}) \qquad (70)$$

we obtain

$$\hat{b}(\vec{r}) = -\frac{2\pi m\hbar c}{e}\hat{\rho}(\vec{r}) \qquad (71)$$

Substituting the density operator $\hat{\rho}(\vec{r})$ by its mean value $n_e=(\nu/2\pi l^2)$ with the fractional integer filling factor ν equal to $\nu=1/m$ with $m\geq 1$, and taking into account the magnetic length $l^2=(\hbar c/eB)$ determined by the external magnetic field B we will find the average value $\bar{b}$ and equality

$$B+\bar{b}=0 \qquad (72)$$

what means that the resulting magnetic field is exactly zero. In this approximation the set of CPs does exist in zero magnetic field. If they are fermions their ground state will be a filled Fermi sea with the Fermi wave vector determined by the magnetic length. If they are bosons, they will undergo the BEC.

In the Section 7 we will discuss the collective elementary excitations above the ground state in the case of CBs on the base of Ginzburg-Landau theory. In Ref. [42] it was shown that applying the mean-field theory one must integrate out the short-distance fluctuations of the

$\psi(r)$ field to obtain an effective action which describe the physics at distance scales larger than the magnetic length. It is supposed that the effective action is of the same form as the microscopic action, but with renormalized stiffness constant, bare mass and the effective interaction strength.

6. QUASI-NAMBU-GOLDSTONE MODES IN THE BOSE-EINSTEIN CONDENSATES

The Goldstone theorem guarantees that the NG modes do not acquire mass at any order of quantum corrections. Nevertheless, sometimes soft modes appear, which are massless in the zeroth order but become massive due to quantum corrections. They were introduced by Weinberg,[5] who showed that these modes emerge if the symmetry of an effective potential of the zeroth order is higher than that of the gauge symmetry and the idea was invoked to account for the emergence of low-mass particles in relativistic physics. Following Ref. [32] now these modes are referred to as quasi-Nambu-Goldstone modes, in spite of the fact that their initial name introduced by Weinberg was pseudo-modes instead of quasi-modes. Georgi and Pais[33] demonstrated that the quasi-NG modes also occur in cases in which the symmetry of the ground state is higher than that of the Hamiltonian.[32] This type of the quasi-Nambu-Goldstone modes is believed to appear, for example, in the weak-coupled limit of A phase of ^{3}He.[37,38]

The authors of Ref. [32] underlined that the spinor BEC are ideal systems to study the physics of the quasi-NG modes, because these systems have a great experimental manipulability and well established microscopic Hamiltonian. It was shown in Ref. [32] that the quasi-NG modes appear in a spin-2 nematic phase. In the nematic condensate, three phases, each of which has a different symmetry, are energetically degenerate to the zeroth order[36] and the zeroth order solution has a rotational symmetry $SO(5)$, whereas the Hamiltonian of the spin-2 condensate has a rotational symmetry $SO(3)$. By applying the Bogoliubov theory of the BEC under the assumption that the $\vec{k}=0$ components of the field operators are macroscopically occupied, it was shown that the order parameter of the nematic phase has an additional parameter independent on the rotational symmetry. The ground state symmetry of the nematic phase at the zeroth order approximation is broken by quantum corrections, thereby making the quasi-NG modes massive. The breaking of the $SO(5)$ symmetry occurs. The number n of the quasi-NG modes was determined by Georgi and Pais[33] in the form of the theorem. It was explained and represented in Ref. [32] as follows:

$$n = \dim(\tilde{M}) - \dim(M) \tag{73}$$

where $\tilde{M}$ is the surface on which the effective potential assumes its minimal values to the zeroth order and $\dim(\tilde{M})$ is the dimension of this surface. The dimension $\dim(M)$ determines the number of the NG modes. This implies that M is a submanifold of $\tilde{M}$ and n is the dimension of the complementary space of M inside $\tilde{M}$.[32]

In the case considered by Goldstone, the dimension of the ring is 1 and the number of the NG modes is 1. This leads to the absence of the quasi-NG modes ($n=0$). Returning to the case of 2D magnetoexcitons in the BEC state with small but nonzero wave vector $\vec{k}$ ($\vec{k}\neq 0$) described by Hamiltonian (16) of Ref. [30], one should remember that both continuous symmetries existing in the initial form (10)[30] were lost. It happened due the presence of the term $\tilde{\eta}(d^\dagger_{\vec{k}}+d_{\vec{k}})$ in the frame of the Bogoliubov theory of quasiaverages. Nevertheless, the energy of the ground state as well as the self-energy parts $\Sigma_{ij}(P,\omega)$, which determine the energy spectrum of the collective elementary excitations depend only on the modulus of the wave vector $\vec{k}$ and do not depend at all on its direction. All these expressions have a rotational symmetry $SO(2)$ in spite of the fact that Hamiltonian (16) of Ref. [30] has lost it. We believe that the condition described by Georgi and Pais[33] may favor the emergence of the quasi-NG modes. We are explaining the existence of the gapped, massive exciton-type branches of the collective elementary excitations obtained in our calculations just by these considerations. These questions will be discussed in Section 11.

7. GINZBURG-LANDAU THEORY FOR THE FRACTIONAL QUANTUM HALL EFFECT

In this section we will follow the collective monograph[43] dedicated to the fractional quantum Hall effect (FQHE), the clear and transparent candidatus scientiarum thesis by Enger[44] and many other papers cited below. The Landau theory of the second order phase transition[45] is based on the introduction of the order parameter, $\phi(rt)$ assuming that the free energy is a regular function of ϕ at least near the critical point. In the case of superconductors and superfluids the role of the order parameter is played by the condensate wave functions. The theory of superconductors was elaborated by Ginzburg and Landau[46] whereas for liquid helium by Ginzburg and Pitaevskii.[47] The microscopical foundations in the latter case were proposed by Pitaevskii[48] and by Gross[49] and can be found in the monograph by Nozieres and Pines.[50] The microscopical theory of superfluidity was firstly proposed by Bogoliubov in the model of weakly interacting Bose gas.[51] The density of the Helmholz free energy $f(r)$ expanded on the small order parameter ϕ has the form

$$f(r) = f_0 + \alpha|\phi|^2 + \frac{\beta}{2}|\phi|^4 + \frac{\hbar^2}{2m}|\nabla\phi|^2 \tag{74}$$

In the case of superconductor it is necessary to include the effect of the applied electromagnetic field which can be done by substituting the canonical momentum $\hat{p}=-i\hbar\nabla$ by the kinetic momentum

$$\vec{p} - \frac{q}{c}\vec{A}(r) \tag{75}$$

where $\vec{A}$ is the vector potential, q is the charge of the Cooper pair, $q = -2e$.

The density of the Gibbs free energy including also the density of the magnetic field energy looks as

$$g(r) = f_0 + \alpha|\phi|^2 + \frac{\beta}{2}|\phi|^4 + \frac{1}{2m}\left|\left(-i\hbar\nabla + \frac{2e}{c}\vec{A}\right)\phi\right|^2 + \frac{B^2}{2\mu_0} \quad (76)$$

where $\vec{B} = rot\vec{A}$. Minimizing the total Gibbs energy $G = \int g(r)dr$ with respect to ϕ and $\vec{A}$ gives

$$\begin{aligned} &\frac{1}{2m}\left(-i\hbar\nabla + \frac{2e}{c}\vec{A}\right)^2\phi + \alpha\phi + \beta|\phi|^2\phi = 0 \\ &\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} = \frac{ie\hbar}{m}(\phi^*\nabla\phi - \phi\nabla\phi^*) - \frac{4e^2}{mc^2}|\phi|^2\vec{A} \end{aligned} \quad (77)$$

This is the Ginzburg-Landau equations, where μ_0 is the magnetic permeability.

The Ginzburg–Pitaevskii–Gross equation for the Bose-Einstein condensate wave function $\phi(r, t)$ is

$$i\hbar\frac{\partial\phi(r, t)}{\partial t} = -\frac{\hbar^2}{2m}\Delta\phi(r, t) + \lambda|\phi(r, t)|^2\phi(r, t) \quad (78)$$

Separating the space and time parts $\phi(r, t) = e^{i\mu t}\phi(r)$, and choosing the chemical potential $\mu = \lambda\rho_0$, one can transform (78) into the equation

$$-\frac{\hbar^2}{2m}\Delta\phi(r) + \lambda(|\phi(r)|^2 - \rho_0)\phi(r) = 0 \quad (79)$$

which is known as Gross-Pitaevskii equation or non-linear Schrodinger equation. As was mentioned in Refs. [51, 52] the GL theory is needed also for the FQHE to better understand this phenomenon.

The FQHE also is a remarkable example of the quantum effects observable on a macroscopic level similarly as superconductivity and superfluidity. All these phenomena have a ground state with non-zero density of particles and in all three cases there are quasiparticle excitations in the form of vortices. But there are some aspects of the FQHE, which are not present in the GL theories of superconductors and superfluids. First of them there is a gap in the spectrum of the collective elementary excitations, which leads to the incompressibility of the FQHE systems. The second important difference is related with the properties of the vortices in the FQHE case. They play the role of the single-particle excitations and have finite creation energy, as opposed to the vortices in the superfluid He-II with an extensive creation energy of the vortex proportional to $\ln(R/a)$, where R is the radius of the system and a is the vortex core.

In addition the FQHE vortices have fractional charges.[52] In numerous papers some variants of the G-L theory for the FQHE were proposed starting with the Lagrangian of the system containing the supplementary term known as Chern-Simons term. It describes the gauge vector potential generated by the vortices; which in their turn are induced by the flux quanta created by the external magnetic field B. Instead of Gibbs free energy the action of the system is studied.

Girvin,[52] and Girvin and MacDonald[53] for the first time proposed a phenomenological variant of the GL theory writing the action S in the form

$$S = \int d^2r\left\{\left|\left(-i\hbar\nabla + \frac{e}{c}A_1(r)\psi(r)\right)\right|^2 + i(\psi^*(r)\psi(r) - n_0) \times \phi(r) - \frac{i\theta}{8\pi^2}(\phi\nabla\times\vec{A}_1 + \vec{A}_1\times\nabla\phi)\right\} \quad (80)$$

where

$$\vec{A}_1 = \vec{A} + \vec{a}; \quad \vec{B} = rot\vec{A} \quad (81)$$

is an effective summary vector-potential composed from the physical external vector-potential $\vec{A}$ generating the magnetic field B, and from a gauge vector-potential $\vec{a}$ created by the vortices. The effective field A_1 represents the frustration arising in the system, when the density of the particles $\rho(r) = |\psi(r)|^2$ deviates away from the quantized Laughlin's density n_0,[42] which determines the fractional filling factor $\nu = 1/m$ with m integer. The density n_0 is named the flux density being determined by the magnetic field B through the magnetic length l in the form $n_0 = 1/m2\pi l^2$, where $l^2 = \hbar c/eB$. The equation of motion for vector $\vec{A}_1$ in a static case is:

$$\theta\nabla\times\vec{A}_1 = (\psi^*\psi - n_0); \quad \theta = 2\pi/m \quad (82)$$

The proposed phenomenological G-L theory allows us to understand that the creation energy of a single vortex is finite and that the vortex has a fractional charge. The difference between the FQHE and ordinary superfluidity was explained by the strong phase fluctuations induced by the frustration. Zhang, Hanson and Kivelson[42] derived their field-theory model starting from the microscopic Hamiltonian. They constructed the G-L theory in a way similar to Girvin but contrary to Girvin in their approach the Chern-Simons term contains only the gauge field $a(r)$.[42] As in the previous papers[53] it was confirmed that the disturbances of the localized density moving the system away from the good filling fractions lead to creation of single-particle excitations. These quasiparticle and quasihole excitations have the form of vortices with static nonuniform finite-energy solutions. Side by side with the single-particle excitations in the Ref. [42] the collective elementary excitations were discussed. For this end the Lagrangian was expanded up to terms quadratic in $\delta\phi$ and δa about the constant solutions corresponding to vacuum expectation values. The fluctuating values $\delta\phi$ and δa were represented in the form of plane waves with vector q. The dispersion relation was found in the form[42]

$$\omega^2(q) = (e\kappa B)^2 + \frac{1}{4}\kappa q^2(\kappa q^2 + 8\lambda n_0) \tag{83}$$

It has a gap in the point $q = 0$ proportional to the external magnetic field B. For negative λ, but for sufficiently small parameter $|\lambda|/\kappa$ the dispersion curve has a roton-type behavior with the same shape as was derived by Girvin, MacDonald and Platzman.[54] The GL theory proposed by Ref. [42] describes the incompressibility, fractional charge and fractional statistics of the quasiparticles. But being a coarse-grained version of the FQHE it makes errors on the magnetic length scale. It treats the gauge field with a mean-field approximation, and reproduces correctly the long-wavelength effects of the quantum Hall systems excluding such details as the description of the vortex core. The idea that the long-wavelength effects of the physical magnetic field are canceled by the gauge field was also suggested by Laughlin[55] and in Ref. [56].

8. POINT VORTICES UNDER THE CONDITIONS OF FQHE

Because the vortices play an important role in the understanding of the FQHE we will provide here more information on this subject. The presentation below beginning with classical hydrodinamics and proceeding to the quantum vortices is given following the paper Ref. [57] and Enger[44] and Myklebust theses.[58] An ideal fluid without viscosity is described in classical hydrodynamics by the continuity equation.

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0 \tag{84}$$

and Euler's equation

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho} \tag{85}$$

where ρ, p and $\vec{v}$ are the density, pressure and velocity field correspondingly in each point of the liquid. The vorticity is defined in 3D hydrodynamics as $\vec{\omega} = \vec{\nabla}\times\vec{v}$. If the liquid is not only ideal but also isoentropic with constant entropy along it, then the vorticity $\vec{\omega}$ obeys to a supplementary continuity equation. The flow is irrotational with a potential flow if $\vec{\omega} = 0$ at all points of the fluid. In this case one can introduce the velocity potential ϕ

$$\vec{v} = \vec{\nabla}\phi; \quad \vec{\omega} = \vec{\nabla}\times\vec{v} = 0 \tag{86}$$

In physical fluids the vorticity is localized in small areas. Outside the vortices most of fluid is irrotational. In a 3D liquid the vortex is a tube with the strength κ defined as

$$\kappa = \int \vec{\omega}\, d\vec{\sigma} = \oint \vec{v}\, d\vec{l} \tag{87}$$

The Helmholtz theorem (also known as Kelvin's circulation theorem) says that in the absence of rotational external forces a fluid that is initially irrotational remains irrotational all the time. In case of 2D fluid the notion of point vortex with zero area is introduced. The velocity field generating such a vortex may be represented by the expressions

$$\begin{gathered}\vec{v} = \frac{\kappa}{2\pi r}\vec{e}_\theta = \frac{\kappa}{2\pi}\left(-\vec{i}\frac{y}{r^2} + \vec{j}\frac{x}{r^2}\right)\\ \vec{e}_\theta = \vec{j}\cos\theta - \vec{i}\sin\theta; \quad \vec{e}_r = \vec{i}\cos\theta + \vec{j}\sin\theta\\ \vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\vec{e}_\theta\end{gathered} \tag{88}$$

Here κ is the vortex strength, whereas the unit vectors $\vec{i}, \vec{j}, \vec{e}_r$ and $\vec{e}_\theta$ corresponds to rectangular and polar 2D coordinates. Following Ref. [41] we must take into account the definition of the curl in the 2D space, namely that the curl of the vector is a scalar and the curl of the scalar is a vector as follows

$$\omega = Curl\vec{v} = \vec{\nabla}\times\vec{v} = \varepsilon^{ij}\partial_i v_j; \quad (CurlS)^i = \varepsilon^{ij}\partial_j S \tag{89}$$

where ε^{ij} is an antisymmetric tensor with the properties

$$\varepsilon^{12} = -\varepsilon^{21} = 1; \quad \varepsilon^{11} = \varepsilon^{22} = 0$$

These rules lead to the vorticity of the point vortex with the velocity field (88)

$$\omega(r) = Curl\vec{v} = \frac{\kappa}{2\pi}\Delta\ln r = \kappa\delta^{(2)}(\vec{r}) \tag{90}$$

The velocity field created by a point vortex has a singularity. It is irrotational or potential almost in all space except of the origin in the point $r = 0$. By this reason the vortex area is zero. Nevertheless the summary vorticity due the singularity (90) is finite. In the same way the circulation of the vortex is also finite as follows

$$\int \omega d^2\vec{r} = \lim_{r\to\infty}\oint \frac{\kappa}{2\pi r}\vec{e}_\theta d\vec{l} = \kappa; \quad d\vec{l} = rd\theta\vec{e}_\theta \tag{91}$$

A fluid containing a point vortex will have potential flow almost everywhere. A point vortex in an incompressible liquid has energy

$$\int \frac{mv^2}{2}d^2\vec{r} = \frac{m\kappa^2}{2\pi}\ln\frac{R}{a} \tag{92}$$

where R is the length scale of the whole system and a is the core radius. A classical system of N point vortices in an incompressible liquid has the kinetic energy associated with each vortex and the interaction energy between them. This interaction does not come from an electric charge of the vortices because they are neutral. For two vortices with guiding centers $\vec{R}_1$ and $\vec{R}_2$ it is useful to define a guiding center of a pair $\vec{R}_{gc}$ and its relative coordinate $\vec{R}_{rel}$ in the form

$$\vec{R}_{gc} = \vec{R}_1 + \vec{R}_2; \quad \vec{R}_{rel} = \vec{R}_1 - \vec{R}_2 \tag{93}$$

REVIEW

The equations of motion for a pair of vortices with equal strengths $\kappa_1 = \kappa_2 = \kappa$ are

$$\dot{X}_{gc} = \dot{Y}_{gc} = 0; \quad \dot{X}_{rel} = -\frac{\kappa Y_{rel}}{\pi R_{rel}^2}; \quad \dot{Y}_{rel} = \frac{\kappa}{\pi}\frac{X_{rel}}{R_{rel}^2} \tag{94}$$

These equations describe a circular motion around a fixed point named as stationary guiding center with an angular velocity Ω depending on the constant separation distance of the vortices $|\vec{R}_{rel}|$ as follows

$$\Omega = \frac{\kappa}{\pi |\vec{R}_{rel}|^2} \tag{95}$$

For a pair of vortices with opposite vorticities $\kappa = \kappa_1 = -\kappa_2$ i.e., for a vortex–antivortex pair the equations of motion are

$$\dot{X}_{gc} = \frac{\kappa}{\pi}\frac{Y_{rel}}{|\vec{R}_{rel}|^2}; \quad \dot{Y}_{gc} = -\frac{\kappa}{\pi}\frac{X_{rel}}{|\vec{R}_{rel}|^2}$$
$$\dot{X}_{rel} = \dot{Y}_{rel} = 0 \tag{96}$$

The vortices will not move relative each other, but will follow a straight line perpendicular to the vector $\vec{R}_{rel}$ connecting the vortices.[44]

This picture is exactly the same as the structure of a 2D magnetoexciton moving with wave vector $\vec{k}$ perpendicular to the vector $\vec{d}$ connecting the electron and hole in the pair with a constant distance $d = kl^2$ at a given $\vec{k}$.

One can remember that the existence of the quantum vortices was suggested for the first time by Onsager,[59] who proposed that the circulation in the superfluid He-II is quantized with the quantum of circulation h/m. The quantum vortices in the He-II were discussed by Feynman,[60] whereas a quantized line was observed by Vinen.[61] The quantization of the vorticity in the He-II can be explained in the frame of GL theory. The velocity field of a superfluid described by the wave function

$$\phi = \sqrt{\rho}e^{iS} \tag{97}$$

can be written as

$$\vec{v} = \frac{\hbar}{m}\vec{\nabla}S \tag{98}$$

The circulation around a close path C becomes

$$\kappa = \oint \vec{v}\, d\vec{l} = \frac{\hbar}{m}\oint \vec{\nabla}S\, d\vec{l} = \frac{\hbar}{m}\delta S \tag{99}$$

δS is the change in the phase of the wave function, as one moves around the close path C. But the wave function must be single valued. By this reason δS must be integer multiple of 2π. It means that

$$\kappa = \frac{\hbar}{m}2\pi s, \quad s = 0, \pm 1, \pm 2, \ldots \tag{100}$$

The vorticity of the quantum vortex has discrete values with the quanum h/m. This definition of vorticity differs from the classical hydrodynamics $\vec{\omega} = \vec{\nabla} \times \vec{v}$. The only rotational invariant wave function having the property (100) being written in polar coordinated has $S = s\theta$

$$\phi(\vec{r}) = f(r)e^{is\theta} \tag{101}$$

It produces the same velocity field as the classical point vortex

$$\vec{v} = \frac{\hbar s}{mr}\vec{e}_\theta = \frac{\kappa}{2\pi r}\vec{e}_\theta \tag{102}$$

The kinetic energy

$$E = \int \frac{1}{2}mv^2 d^2 r = \frac{\hbar^2 \pi}{m}s^2 \ln(R/\xi) \tag{103}$$

is now expressed through the coherence length ξ instead of the core radius a. The cutoff at ξ is used to avoid the logarithmic divergence neared the vortex core.

Inserting the vortex function (101) into the Ginzburg–Pitaevskii–Gross equation, Myklebust[44,58] found the following equation for the function $f(r)$

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \left(2 - \frac{s^2}{r^2}\right)f - 2f^3 = 0 \tag{104}$$

It depends only on s^2. Contrary to the He-II, the Bose-Einstein condensate in superconductors is formed by the Cooper pairs with the charge $q = -2e$ instead of the neutral atoms. The type II superconductors allow the magnetic field to penetrate in metals forming quantized vortices, while in the type-I superconductors the magnetic field cannot penetrate.[62] The quantized vortices exist in the form of filaments named Abrikosov's lines. They have a mixed electron and electromagnetic field origin and were described for the first time by Abrikosov[63] on the base of the G-L theory with nonzero electromagnetic field $\vec{A}$. It was shown that the magnetic flux through the vortex tube is quantized with the flux quantum ϕ_0

$$\int \vec{B}d\vec{\sigma} = \oint \vec{A}d\vec{l} = n\phi_0; \quad \phi_0 = \frac{2\pi\hbar c}{|q|} \tag{105}$$

The total energy per a unit length of the vortex tube is finite and equals to

$$E = \left(\frac{\phi_0}{4\pi\lambda}\right)^2 \ln\frac{\lambda}{\xi}; \quad \lambda > \xi \tag{106}$$

where λ is the penetration length of the magnetic field into the II-type superconductors as was introduced by London and London[64] and ξ is the correlation length between the electrons in the Cooper pair. Girvin[52] suggested that the contribution of the electromagnetic field in the resultant current density $\vec{j}(\vec{r})$ determined in the case of FQHE as

$$\vec{j}(\vec{r}) = \frac{1}{2}\{\psi^*(r)(-i\hbar\vec{\nabla})\psi(r) + \psi(r)(i\hbar\vec{\nabla})\psi^*(r)\} + \frac{e}{c}\vec{A}\psi^*(r)\psi(r) \tag{107}$$

reorganizes the point vortex state in such a way that its resultant circulation at great distance $r \to \infty$ will be zero

$$\oint \vec{j}(r)\, d\vec{l} = 0 \tag{108}$$

It is possibly only for the condition when the magnetic flux through the vortex surface is quantized in the form

$$\int rot\vec{A}\, d^2\vec{r} = \oint \vec{A}\, d\vec{l} = -m\phi_0; \quad \phi_0 = \frac{2\pi\hbar c}{|e|} \tag{109}$$

This value being multiplied by $n_0|e|/c$ compensates exactly the circulation arising from the electron part of the current density

$$\oint \frac{1}{2}\{\psi^*(r)(-i\hbar\vec{\nabla})\psi(r) + \psi(r)(i\hbar\vec{\nabla})\psi^*(r)\}\, d\vec{l} = 2\pi\hbar m n_0 \tag{110}$$

because the wave function $\psi(r)$ has the form

$$\psi(r) = \sqrt{n_0} f(r) e^{im\theta}; \quad f(r) \to 1; \quad r \to \infty \tag{111}$$

The number of magnetic flux quanta $-m$ must be opposite to the magnetic quantum number of the electron wave function. The creation energy of such point vortex is finite and no extensive as in the case of a pure electron vortex. As was mentioned by Girvin and MacDonald[53] the isolated vortex cost only a finite energy. They can be excited thermally by one. Earlier it was necessary to create a pair vortex-antivortex with finite creation energy for a pair as a whole, but with extensive energy for each of them. Only in the last case the Kosterlitz-Thoulless phase transition was possible being related with the unbinding of the vortices in the pairs.

9. GAUGE TRANSFORMATIONS AND STATISTICAL GAUGE FIELD

Girvin and MacDonald[53] revealed a hidden symmetry of the Laughlin's[65] ground state wave function describing the FQHE of the 2D one-component electron gas (OCEG). This wave function is

$$\psi(z_1, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^m \exp\left[-\frac{1}{4}\sum_k |z_k|^2\right] \tag{112}$$

The filling factor of the lowest Landau level (LLL) is a fractional integer $\nu = 1/m$, with integer $m > 1$. $z_k = x_k + iy_k$ are the complex coordinates of the particles in symmetric gauge. With respect to the interchanging of any two particles the wave function (112) is anti-symmetric at odd values of m and symmetric at even values, describing the fermions and bosons, correspondingly. But changing the phase of the wave function (111) using a singular gauge transformation

$$\begin{aligned}\psi_{new}(z_1, \ldots, z_N) &= \exp[-im\sum_{i<j}\arg(z_i - z_j)]\psi(z_1, \ldots, z_N) \\ &= \prod_{i<j}|z_i - z_j|^m \exp\left[-\frac{1}{4}\sum_k |z_k|^2\right]\end{aligned} \tag{113}$$

we have obtained a bosonic type wave function at any integer values of $m > 1$. The off-diagonal matrix elements of the density matrix $\rho(z, z')$ calculated with the function (112) are short-ranged with a characteristic scale given by the magnetic length, while those calculated with the wave function (113) $\tilde{\rho}(z, z')$ have a slowly decreasing behavior with a power law $|z - z'|^{m/2}$. The singular gauge density matrix $\tilde{\rho}(z, z')$ has an off-diagonal long-range order (ODLRO). The physical origin of this difference is related to the presence of the vortices induced around each particle under the influence of the magnetic flux quanta, as was explained by Stormer.[66]

The presence of the vortices can be demonstrated using more simple example proposed by Enger[44] with a wave function $\psi(z)$ of two particles depending only on their relative coordinate z. It is supposed that $\psi(z)$ obeys to any statistics and after the particle interchanging it becomes

$$\begin{aligned}&\psi(e^{i\pi}z) = e^{i\theta}\psi(z); \quad \theta = \pi(2n+1) \text{ for fermions} \\ &n = 0, \pm 1, \pm 2\ldots \quad \theta = 2\pi n \text{ for bosons}\end{aligned} \tag{114}$$

A gauge transformation $e^{i\eta(z)}$ transforms the wave function $\psi(z)$ into another bosonic type wave function

$$e^{i\eta(z)}\psi(z) = \phi(z) = \phi(e^{i\pi}z) \tag{115}$$

To satisfy this requirement and the equalities

$$e^{i\eta(e^{i\pi}z)}\psi(e^{i\pi}z) = e^{i\eta(z)}\psi(z) = e^{i\eta(e^{i\pi}z)}e^{i\theta}\psi(z) \tag{116}$$

the function $\eta(z)$ must satisfy the equation

$$\begin{aligned}&\eta(z) - e^{i\pi}z = \theta \\ &\eta(z) = -\frac{\theta}{\pi}\arg z = -\frac{\theta}{\pi}\arctan\frac{y}{x}\end{aligned} \tag{117}$$

The transformation of the wave function (115) must be accompanied by the transformation of the electromagnetic field potentials A_μ[44]

$$\frac{e}{\hbar c}A_\mu \to \frac{e}{\hbar c}A_\mu + \partial_\mu\eta(z) = \frac{e}{\hbar c}(A_\mu + a_\mu) \qquad \mu = 0, 1, 2 \tag{118}$$

In such a way side by side with the electromagnetic potential A_μ supplementary gauge potential a_μ created by the vortices appears.

$$\begin{aligned}&\frac{e}{\hbar c}a_\mu(\vec{r}) = \partial_\mu\eta(\vec{r}) \\ &a_\mu(\vec{r}) = \frac{\hbar c}{e}\partial_\mu\eta(\vec{r}) = -\frac{\hbar c\theta}{\pi e}\partial_\mu \arctan\frac{y}{x}\end{aligned} \tag{119}$$

The statistical gauge vector potential has the expression

$$\begin{aligned}&\vec{a} = \frac{\hbar c\theta}{\pi e}Curl\ln r = \frac{\hbar c\theta}{\pi e}\vec{\nabla}\times\ln r \\ &a_i = \frac{\hbar c\theta}{\pi e}\varepsilon^{ij}\partial_j\ln r; \quad i, j = 1, 2\end{aligned} \tag{120}$$

This vector potential is created by the vortices arising near each particle. It leads to magnetic field strength[41]

$$b(r) = Curl\vec{a}(r) = \vec{\nabla}\times\vec{a}(r) = \varepsilon^{ij}\partial_i a_j = -\frac{\hbar c\theta}{e\pi} \times \Delta\ln r = -\frac{2\hbar c\theta}{e}\delta^2(\vec{r}); \quad \frac{\Delta\ln r}{2\pi} = \delta^2(\vec{r}) \tag{121}$$

The magnetic flux created by this magnetic field is

$$\int b(r)d^2\vec{r} = -\frac{2\pi\hbar c\theta}{e} = -\frac{\theta}{\pi}\phi_0; \quad \phi_0 = \frac{hc}{e} \tag{122}$$

It equals to $-(2n+1)\phi_0$, when the initial particles described by the function $\psi(z)$ were fermions, and equals to $-2n\phi_0$ for boson wave function $\psi(z)$. This result shows that the initial fermion particles each of them attaching an odd number of flux quanta transform themselves into a composite bosons described by the new wave function $\phi(z)$ which obeys to Bose statistics. The effective mass m and charge e remain the same at least in the given approximation but the composition and statistics of the final quasiparticles are changed. It is said that the electron attached an odd number of flux tubes, though in reality such tubes do not exist. We can say that in our case the initial particles are fermions or electrons whereas the final quasiparticles are bosons. Formula (121) may be generalized for any number of particles, which create in a common way the resultant magnetic field

$$b(r) = -\frac{2\hbar c\theta}{e}\sum_{i=1}^{N}\delta^2(\vec{r}-\vec{r}_i) = -\frac{2\theta\hbar c}{e}\rho(\vec{r}) \tag{123}$$

where $\rho(\vec{r})$ is the density of the particles.

As was mentioned above, Zhang, Hanson and Kivelson[42] have generalized the Ginzburg-Landau theory introducing into the Lagrangian a supplementary Chern-Simoms[35] term related with the influence of the statistical gauge field. The Lagrangian of the Ginzburg–Landau–Chern–Simons (GLCS) theory in the form presented by Enger[44] looks as

$$L = i\hbar\phi^*\left(\partial_t + \frac{ie}{\hbar c}(A_0+a_0)\right)\phi + \frac{\hbar^2}{2m}\phi^*\left(\vec{\nabla}+\frac{ie}{\hbar c}(\vec{A}+\vec{a})\right)^2 \times\phi - \frac{\lambda}{2}(\phi^*\phi-\rho_0)^2 + \frac{\mu}{2}e^{a}_{\mu}\partial_\nu a_\sigma \tag{124}$$

Here the following denotations were used: $\mu,\nu,\sigma = 0,1,2;$ $\partial_0 = \partial_t,$ $\partial_i = \{\partial_1 = \partial_x, \partial_2 = \partial_y\}$. The tensor $\varepsilon^{\mu\nu\sigma}$ has the nonzero components only for different values of μ,ν,σ. They change the signs at any permutations of two indexes as follows:

$$\varepsilon^{012} = 1, \quad \varepsilon^{021} = -1, \quad \varepsilon^{102} = -1, \quad \varepsilon^{120} = 1 \text{ etc} \tag{125}$$

The external electromagnetic 2D vector-potential $\vec{A}$ and the scalar potential A_0 are taken as $A_\mu = (A_0, \vec{A})$. a_μ is the statistical gauge potential with three components. Two of them $\vec{a} = (a_1, a_2)$ generate the statistical "magnetic" field and the third component a_0 gives rise to the statistical "electric" field. Two parameters m and e of the Lagrangian (124) are the effective mass and charge of the final type quasiparticles obeying the Bose statistics. They can differ from the mass and charge of the initial particles. The λ and ρ_0 parameters are typical for the G-L theory, while μ is the Chern-Simons parameter.[67] Variations of (124) with respect to ϕ^* leads to the nonlinear Shrodinger equation:

$$\left[i\hbar\partial_t - \frac{e}{c}(A_0+a_0)\right]\phi = -\frac{\hbar^2}{2m}\left[\vec{\nabla}+\frac{ie}{\hbar c}(\vec{A}+\vec{a})\right]^2\phi - \lambda(\phi^*\phi-\rho_0)\phi \tag{126}$$

The variation of (124) with respect to a_0 gives

$$\mu\varepsilon^{ij}\partial_i a_j = e\phi^*\phi = e\rho \tag{127}$$

which can be transcribed as

$$\mu Curl\vec{a} = \mu\vec{\nabla}\times\vec{a} = \mu b = e\rho \tag{128}$$

Parameter μ can be determined comparing it with the expression (119)

$$\mu = -\frac{2\theta\hbar c}{e^2} \tag{129}$$

For the initial fermion particles with $\theta = \pi(2n+1)$ the parameter μ of the Lagrangian (124) equals to $-(2\pi\hbar c/e^2)(2n+1) = -(2n+1)(\phi_0/e)$. The third variation with respect to a_i gives

$$\mu\varepsilon^{ij}(\vec{\nabla}a_0 - \partial_t\vec{a}) = e\vec{j}(r) \tag{130}$$

where $\vec{j}(r)$ is the current density 2D vector determined as

$$\vec{j} = \frac{\hbar}{2mi}\left\{\phi^*\left(\vec{\nabla}-\frac{ie}{\hbar c}(\vec{A}+\vec{a})\right)\phi - \phi\left(\vec{\nabla}+\frac{ie}{\hbar c}(\vec{A}+\vec{a})\right)\phi^*\right\} \tag{131}$$

The Eq. (130) states that the statistical "electric" field

$$\vec{\varepsilon} = -\vec{\nabla}a_0 + \frac{1}{c}\frac{\partial}{\partial t}\vec{a} \tag{132}$$

is related with the particle current density $\vec{j}(r)$. The energy density of the GLCS system in the purely static external magnetic field ($A_0 = 0$) equals to

$$E = \frac{\hbar^2}{2m}\left|\left(\vec{\nabla}-\frac{ie}{\hbar c}(\vec{A}+\vec{a})\right)\phi\right|^2 + \frac{\lambda}{2}(\rho-\rho_0)^2 \tag{133}$$

A simple solution to these equations can be obtained by setting

$$\phi(\vec{r}) = \sqrt{\rho_0}e^{iS(\vec{r})} \tag{134}$$

It must obey the equation

$$\vec{\nabla}S + \frac{e}{\hbar c}(\vec{A}+\vec{a}) = 0 \tag{135}$$

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In the case $S = Const$ we have

$$\vec{A} + \vec{a} = 0 \tag{136}$$

It means that the corresponding magnetic fields $B = \vec{\nabla} \times \vec{A}$ and $b = \vec{\nabla} \times \vec{a}$ cancel each other. The final quasiparticles named as composite bosons[68–70] do not feel the net magnetic field and behave as bosons interacting with eachother via the ϕ^4 type interaction. The notion of composite particles (CPs) consisting from electrons and attached magnetic flux quanta was introduced firstly by Wilczek.[68]

As was mentioned above, the collective elementary excitations of the described ground state are the plane waves. Their dispersion law has a gap, and this means that the system is an incompressible quantum liquid, which can not be excited by a very small perturbation.[69, 70]

Above we have discussed the case when the initial wave function $\psi(z)$ with Fermi statistics was transformed into another wave function $\phi(z)$ obeying Bose statistics using a singular gauge transformation. Read[40, 71–73] investigated the system of 2D charged bosons interacting with a transverse magnetic field and between themselves. The filling factor of the LLL was supposed to be one. It means that there is one flux quantum for each particle. Following Read[40, 71–73] it is equivalently to say that there exists one vortex for each particle. In this case the vortex has the charge of opposite sign in comparison with the charged boson and the Fermi statistics. Now the gauge transformation attaching one vortex to each charged boson will create composite particles with resulting charge zero and with Fermi statistics. The neutral composite fermions (CFs) will move in zero magnetic field. Such system can be described in the frame of the Fermi-liquid-theory. Another variant proposed by Halperin, Lee and Read[39] was considered in Section 5. The starting Hamiltonian describes the electrons forming a 2D electron gas (2DEG) with filling factor ν of the LLL equal to one half ($\nu = 1/2$). Now for each electron there are two flux quanta or two vortices each of them having the charge $-e/2$ and Fermi statistics. Two vortices are equivalent to one 2-fold vortex with charge $-e$ and Bose statistics. The gauge transformation of the wave function will transform the initial charged electrons into the composite neutral fermions each of them consisting from one electron and 2-fold vortex. The Hamiltonian of the system will be changed because side by side with the external magnetic field will appear a supplementary gauge magnetic field, which in well definite conditions cancels exactly the external magnetic field. The initial charged fermions were converted in neutral CFs moving in a zero resulting magnetic field. The fictitious Chern-Simons "magnetic" field created by the vortices being averaged in the mean-field approximation cancels exactly the external magnetic field only in statistical sense and under the definite conditions. It happens when the mean density of the electrons corresponds to the fractional integer filling factor. In the present example with $\nu = 1/2$ the gauge transformation does not modify the statistics of the composite particles (CPs). As earlier, they are neutral CFs in a zero magnetic field. The singular gauge transformations were firstly introduced by Wilczek.[68]

The single-particle elementary excitations appear in the form of the fractionally charged vortices. They are fermions and have finite creation energy as was underlined by Girvin,[52] and Girvin and MacDonald.[53] Read[39, 40, 71–73] argued that the ground states of the systems in the condition of FQHE with different fractional integer filling factors $\nu = 1/m$ with $m = 1, 2, 3, \ldots$ contain electrons bound to vortices, since such binding lowers the system's energy. A m-fold vortex carriers a charge $-e\nu m$ in the fluid, where e is the electron charge $e = -|e|$. The electron-m-vortex composite, named as CP, at $\nu = 1/m$ has a net charge zero and behaves like a particle in a zero magnetic field. The vortex is sensitive to the density of electrons, which can vary in space and time even when the external magnetic field and the average filling factor are fixed. The m-fold vortices are fermions for m odd and bosons for m even. The composite boson particles can undergo the Bose-Einstein condensation (BEC), because it minimizes their "kinetic" energy. Just the BEC of CBs is the interpretation of the Laughlin's states.[65] The origin of the "kinetic" energy is the potential energy of the interaction between the particles. In the case of electrons it is the Coulomb electron–electron interaction which is not canceled by the gauge transformation and CS gauge potential. It is named as "kinetic" because it depends on the wave vector of the operators.[39] The bound objects such as CPs do, in fact, have such an effective "kinetic" energy. There is an attraction between an electron and m-fold vortex. It plays for the electron the role of a correlation quasihole. As was shown in Ref. [39] the CPs may exist in the form of plane waves and the many-particle wave functions also can be characterized by the wave vector $\vec{k}$.

The creation operator in the coordinate representation (56) can be rewritten in momentum representation as follows

$$\psi^{\dagger}(\vec{k}) = \int d^2\vec{r} e^{i\vec{k}\vec{r}} \psi^{\dagger}(\vec{r}) \tag{137}$$

A CP with $\vec{k} = 0$ would have the electron exactly at the zeroes of the wave function or in the center of the vortex, whereas the CP with wave vector $|k| \neq 0$ has the electron displaced by the distance $|k|l^2$ from their center. One can say that the electron and its correlation quasihole or in another words the electron and the m-fold vortex experience a potential $V(|k|)$ due to the Coulomb interaction of the electron with other electrons excluded from the vortex core. All these interactions take place in the presence of the neutralizing background. The electron and the m-fold vortex experience the magnetic field of the same strength. Both components of the pair drift in the same direction perpendicular to the vector connecting their centers, so that their separation remains constant and equal to $|k|l^2$.

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The energy of a pair is $V(|k|)$ and its group velocity is $\partial V(|k|)/\partial |k|$.[40, 71–73] We can add that this picture coincides with the structure of the 2D magnetoexciton, where the energy $V(|k|)$ equals to the expression $E(|k|)$[39]

$$E(|k|) = 2\sum_{Q} W_Q \mathrm{Sin}^2\left(\frac{[\vec{Q}\times\vec{k}]_z l^2}{2}\right)$$
$$W_Q = \frac{2\pi e^2}{\varepsilon_0 S|\vec{Q}|} e^{2_l 2}/2 \qquad (138)$$

Here ε_0 is the dielectric constant and S is the layer surface area.

Girvin, MacDonald and Platzman[54] elaborated the theory of the collective elementary excitation spectrum in the case of FQHE, which is closely analogous to the Feynman's theory of superfluid helium. The predicted spectrum has a gap at $k = 0$ and a deep magneto-roton minimum at finite wavevector, which is a precursor to the gap collapse associated with Wigner crystal instability. They supposed the existence of only one branch of the collective elementary excitations spectrum. In this approximation named as single mode approximation (SMA) they have constructed the wave functions of the excited states ϕ_k acting with the operator of the particle density $\hat{\rho}_k$ on the ground state wave function ψ_g in the form $\phi_k = \hat{\rho}_k\psi_g$. They determined the energy of the excited state $\Delta(k)$ as

$$\Delta(k) = \frac{\langle\phi_k|(H-E_0)|\phi_k\rangle}{\langle\phi_k|\phi_k\rangle} = \frac{\langle\psi_g|P_k^\dagger[H_0, P_k]|\psi_g\rangle}{\langle\psi_g|P_k^\dagger P_k|\psi_g\rangle}$$
$$= \frac{f(k)}{s(k)} \qquad (139)$$

where $f(k)$ is the oscillator strength and $s(k)$ is the static structure factor. The total oscillator strength sum is saturated by the cyclotron mode contribution, and $f(k)$ has a dependence of the type $|k|^4$. As was established by Lee and Zhang[74] the influence and the contribution of the quantum vortices to the dynamical and static structure factors is important. It leads to dependence $s(k) \sim |k|^2$ at $k \to 0$. In this case $\Delta(k)$ has a gap. Neglecting the influence of the quantum vortices the dependence $s(k)$ is proportional to $|k|^2$ and the energy spectrum is gapless $\Delta(k) \approx k^2$ at $k \to 0$ as a Goldstone mode.[2] In conclusion, for the FQHE in the case $\nu < 1$ with fractionally filled Landau level the Pauli principle no longer excludes the low-lying intra-Landau-level excitations. They exist side by side with the inter-Landau-level excitations. The last excitations have a cyclotron energy gap.[75]

We are studying a coplanar electron–hole (e–h) system with electrons in conduction band and with holes in valence band in a strong perpendicular magnetic field. Previously such system has been studied in a series of papers Refs. [9, 22–25, 27, 76]. Most of them were dedicated to the theory of 2D coplanar magnetoexcitons. All the same, there were papers dedicate to another aspects of these systems. For example MacDonald, Rezayi and Keller[77] as well as Joglekar and MacDonald[78] have discussed the photoluminescence (PL) spectrum of the coplanar system in the FQHE regime. It was mentioned that the PL spectrum does not exhibit anomalies associated with the FQHE. However when the electron and hole layers were separated a new peak in the PL spectrum appears, when the filling factor exceeds a fraction ν_0 at which an incompressible quantum liquid occurs. The new peak is separated from the main spectral features by the quasiparticle–quasihole gap. We are interested in the distribution of the flux quanta in the case of e–h system with equal average numbers of electrons and holes $\bar{N}_e = \bar{N}_h$ with the filling factor $\nu = \bar{N}_e/N$, where N is the total number of flux quanta $N = S/2\pi l^2$, where S is the layer surface area and $2\pi l^2$ is the area of the cyclotron orbit. For the fractional integer filling factor there are an integer number of flux quanta per each e–h pair. The creation of the vortices in this case is not studied at present time. But one can expect that in the case of magnetoexcitons vortices will be neutral, whereas in the case of pure electron and pure hole vortices their "magnetic" gauge fields will compensate each other, so that the charge, the statistics of the particles and the external magnetic field will remain the same in the mean-field approximation with equal densities of electrons and holes. Nevertheless due to quantum fluctuations and the deviations in space and time of the electron and hole densities from their average values one can expect the influence of the pure electron and hole quantum vortices on the physics of magnetoexcitons side by side with the influence of the neutral quantum vortices formed by the magnetoexcitons themselves. The last quantum vortices determine the Berezinskii–Kosterlitz–Thouless phase transition.[79, 80]

10. QUANTUM HALL EXCITONS IN BILAYER ELECTRON SYSTEMS

In this section we give a short review of the Bose-Einstein condensation (BEC) of the quantum Hall excitons (QHExs) arising in the bilayer electron systems under the conditions of the quantum Hall effect (QHE) at one half filling factor $\nu = 1/2$ for each layer and the total filling factor for two layers equal to unity $\nu_t = 1$. The purpose is to compare this phenomenon with the case of BEC of two-dimensional (2D) magnetoexcitons. Such comparison will give a better understanding of the underlying physics and allows to verify the accuracy of the made approximations.

In the Ref. [81] Fertig investigated the energy spectrum of a bilayer electron systems in a strong perpendicular magnetic field and introduced the concept of the interlayer phase coherence of the electron states in two adjacent layers, which leads to the model of quantum Hall excitons under the condition of their BEC. Unexpectedly a strong evidence of exciton BEC was ultimately found in such surprising system as a double layer 2D electron system at a high magnetic field.[82] In the QHE regime the excitons consist of electrons in the lowest Landau level (LLL) of the conduction band of one layer being bound to the holes which appear in the LLL of the conduction band in another layer. The formation of such unusual holes is due to the possibility to consider the half-filled LLL by electrons of the conduction band, for example, of the first layer as being completely filled by electrons with filling factor $\nu = 1$ and simultaneously being half-filling by holes in the same conduction band. The full-filling electrons of the first layer are considered as being compensated by the impurity doped adjacent layer and the theoretical model takes into account only the holes in the first layer and the electrons in the second layer. Both components belong to the LLLs of the same conduction band and are characterized by a half-filling factor for each of them. This new type of excitons named QHExs appears whenever the temperature and the layer separation are small enough and the total density n_t of electrons in the double layer system equals to the degeneracy $eB/hc = n_t = 1/2\pi l^2$ where l is the magnetic length. The total filling factor $\nu_t = n_t 2\pi l^2$ equals unity. The new collective electronic state introduced by Fertig[81] exhibits several dramatic electrical transport properties revealed in Ref. [83–85]. As was mentioned in REf. [82] the BEC of the QHExs reflects the spontaneously broken $U(1)$ symmetry in which the electrons are no-longer confined to one layer or to the other, but instead they reside in a coherent linear combination of the two layers. This interlayer phase coherence develops only when the effective interlayer separation d/l is less than a critical value $(d/l)_c$. At large d/l the bilayer system behaves qualitatively like the independent 2D electron systems. Following[86] this new state can be distinguished as a Fermi liquid state of composite fermions. It is unique because unlike other QH states it possesses a broken symmetry in the absence of the interlayer tunneling. It can be viewed as a pseudospin ferromagnet with the pseudospin encoding the layer degrees of freedom or as an exciton BEC with QHExs formed from electrons and holes confined to different layers.

There are two energy scales in the double-layer systems. One is the potential energy V between the electrons in different layers. The second is the energy gap Δ_{SAS} between the symmetric and asymmetric states of electrons in two layers measuring the tunneling amplitude between them. The capability of tuning the strength of the interlayer interaction by changing the gate voltage provides the opportunity to explore the $\nu_t = 1$ system through its transformation between the weak and strong interaction limits and to study the phase transitions between the compressible Fermi liquid and the incompressible QH states as a function of d/l.[86] In the most theoretical investigations of the QHExs, except the paper Ref. [86], the simplifying assumption of the fully spin polarized electrons was used. Below, in our discussions the Zeeman energy will be not included. Following the Ref. [87], in the absence of the interlayer tunneling there are two $U(1)$ symmetries. One is associated with the conservation of the total electric charge $N_1 + N_2$, where N_1 and N_2 are the numbers of electrons in two layers, and the other is related with the conservation law of $N_1 - N_2$. For these conditions the gapless mode appears. It is the Nambu-Goldstone (NG) mode arising from the broken $U(1)$ symmetry associated with $N_1 - N_2$ and characterized by the off-diagonal long-range order in the tunneling operator $a_{1p}^{\dagger} a_{2q}$, where a_{1p} and a_{2q} are the electron annihilation operators in the LLLs of the conduction band of two layers.

Within the mean-field effective theory the appearance of the gapless mode may be attributed to the coherent fluctuation of the electron flux and density describing the relative fluctuations of the electron densities in two layers. At finite interlayer tunneling the number $N_1 - N_2$ is no-longer conserved. As well, the currents in each layer are no-longer separately conserved.

The collective excitation spectrum of the two-layer electron system with $\nu_t = 1$ was investigated by Fertig[81] on the basis of the theoretical model without tunneling but with different interlayer separation including $d = 0$ and taking into account that at $d > 0$ the Coulomb interlayer electron–electron interaction is smaller than the intralayer interaction. The ground state wave function proposed by Fertig[81] introduces the interlayer phase coherence reflecting a new state, in which the electrons are no-longer confined to one layer or to another, but instead they reside in coherent linear combinations of the two layer states as follows

$$|\psi\rangle = \prod_t (u a_{1t}^{\dagger} + v a_{2t}^{\dagger})|0\rangle, \quad u^2 + v^2 = 1 \tag{140}$$

The lowest levels of the Landau quantization in the Landau gauge are characterized by the quantum number $n = 0$ and the uni-dimensional wave number t, with $|0\rangle$ being the vacuum state. The equality $u^2 = v^2 = 1/2$ reflects the half-filling of the LLL in each layer. Introducing the hole operator $d_t^{\dagger}, d_t$ for the first layer instead of the operators $a_{1t}^{\dagger}$ and a_{1t} the function (144) was transcribed in the form

$$\begin{gathered} |\psi\rangle = \prod_t (u + v a_t^{\dagger} d_{-t}^{\dagger})|\psi_0\rangle, \quad |\psi_0\rangle = \prod_t a_{1t}^{\dagger}|0\rangle \\ a_{2t} = a_t, \quad a_{2t}^{\dagger} = a_t^{\dagger}, \quad a_{1t} = d_{-t}^{\dagger}, \quad a_1 t^{\dagger} = d_{-t} \end{gathered} \tag{141}$$

The operators $a_t^{\dagger} d_{-t}^{\dagger}$ create the electron–hole pairs with total wave vector equal to zero. The wave function (141) can be interpreted as describing the BEC of the QHExs. This model is similar to the case of BEC of 2D magnetoexcitons studied in Refs. [22–24, 76]. In the last case the holes were formed in the frame of the valence band. The valley-density two-particle integral operators introduced in

Ref. [81] in the electron–hole representation are

$$\begin{aligned}\rho^{\pm}(q) &= \sum_t e^{iq_y t l^2}[d^{\dagger}_{-q_x/2-t} d_{q_x/2-t} + a^{\dagger}_{-q_x/2+t} a_{q_x/2+t} \\ &\quad \pm (d_{q_x/2-t} a_{q_x/2+t} - a^{\dagger}_{-q_x/2+t} d^{\dagger}_{-q_x/2-t})] \\ \rho_Z(q) &= \sum_t e^{iq_y t l^2}(d_{q_x/2-t} a_{q_x/2+t} + a^{\dagger}_{-q_x/2+t} d^{\dagger}_{-q_x/2-t}) \\ \rho_F(q) &= \sum_t e^{iq_y t l^2}(a^{\dagger}_{-q_x/2+t} a_{q_x/2+t} - d^{\dagger}_{-q_x/2-t} d_{q_x/2-t})\end{aligned} \tag{142}$$

We introduce our designations for the exciton, optical and acoustical plasmon operators with holes in the conduction band, as follows

$$\begin{aligned}\rho(q) &= \sum_t e^{iq_y t l^2}[d^{\dagger}_{-q_x/2-t} d_{q_x/2-t} + a^{\dagger}_{-q_x/2+t} a_{q_x/2+t}] \\ D(q) &= \sum_t e^{iq_y t l^2}[a^{\dagger}_{-q_x/2+t} a_{q_x/2+t} - d^{\dagger}_{-q_x/2-t} d_{q_x/2-t}] \\ d(q) &= \frac{1}{\sqrt{N}} \sum_t e^{iq_y t l^2} d_{q_x/2-t} a_{q_x/2+t} \\ d^{\dagger}(q) &= \frac{1}{\sqrt{N}} \sum_t e^{iq_y t l^2} a^{\dagger}_{q_x/2+t} d^{\dagger}_{q_x/2-t}\end{aligned} \tag{143}$$

In the case of holes in the valence band there are opposite signs in the expressions for $\rho(q)$ and $D(q)$.

The relations between two sets of operators are

$$\begin{aligned}\rho^{\pm}(q) &= \rho(q) \pm \sqrt{N}(d(q) - d^{\dagger}(-q)) \\ \rho_z(q) &= \sqrt{N}(d(q) + d^{\dagger}(-q)) \\ \rho_F(q) &= D(q)\end{aligned} \tag{144}$$

The response functions were introduced as follows[81]

$$\begin{aligned}\chi_{\pm}(q,\omega) &= -i\int_0^{\infty} dt e^{i\omega t}\langle[\rho^{\mp}(q,t), \rho^{\pm}(-q,0)]\rangle \\ \chi_z(q,\omega) &= -i\int_0^{\infty} dt e^{i\omega t}\langle[\rho_z(q,t), \rho_z(-q,0)]\rangle \\ \chi_F(q,\omega) &= -i\int_0^{\infty} dt e^{i\omega t}\langle[\rho_F(q,t), \rho_F(-q,0)]\rangle\end{aligned} \tag{145}$$

The poles of these functions represent the excitations of the system. The excitations may be thought as a valley-density waves or pseudospin density waves. The calculation of the response functions were effectuated by Fertig[81] using the diagrammatic expansion elaborated by Kallin and Halperin.[75] The response functions were written in terms of vertex functions, Green's functions and self-energy parts. This approximation is shown by the diagrams in Figure 3(a) Ref. [81] neglecting the diagrams which contain supplementary the *e–h* bubbles. Their contribution is negligible only when the excited Landau levels (ELLs) are taken into account and the bubbles have an energy denominator $\hbar\omega_c$, where ω_c is the cyclotron frequency increasing in the strong field limit. A self-consistent calculation of the vertex function including the bubbles is quite difficult. Below we will present the results obtained in Ref. [81] for the energy spectrum of the collective elementary excitations.

For $d = 0$ the interaction Hamiltonian is invariant under the unitary transformation $SU(2)$. The specific choice of the ground state wave function (141) is a broken symmetry state and one expects the appearance of a NG mode. At $d = 0$ the NG mode has a dispersion relation $\omega(k) \sim k^2$ for the small k. For $d > 0$ the problem can be mapped onto an equivalent spin system with linear dispersion relation at small wave vectors. The NG mode at $d > 0$ has a linear dispersion law with a slope dependence on d, which is similar with that of the acoustical plasmon mode of a two-layer system in the absence of the magnetic field.[88] To better understand this result one may recall the BEC interpretation of the ground state wave function. Indeed. at $d > 0$ the inter-layer electron–hole Coulomb attraction is smaller than the intralayer electron–electron and hole–hole repulsions, that leads to a resultant repulsion in the system and to the transformation of the parabolic dispersion law into the linear one at small values of wave vectors as in the Bogoliubov theory of superfluidity of the Bose gas.[1] At kl of the order unity the dispersion law of Ref. [81] develops a dip at certain critical values of d, indicating that the system trends to undergo a phase transition.

Another considerations concerning the gapless modes in the FQHE of multicomponent fermions can be found in Refs. [89,90]. The above branch of the energy spectrum corresponds to the response function $\chi_{\pm}(q,\omega)$. The operators $\rho^{\pm}(q)$ describe two superpositions of the optical plasmon and exciton mode operators. There are two other operators $\rho_z(q)$ and $\rho_F(q)$ which describe the pure exciton modes and the acoustical plasmon mode. As was established in Ref. [81] the last two response functions $\chi_z(q,\omega)$ and $\chi_F(q,\omega)$ have no poles when the excitations in the frame of the LLLs are considered. The excitations associated with these functions are considered to be higher in energy than the NG mode discussed above by an amount of energy of the order $\hbar\omega_c$. It means that the pure exciton and acoustical plasmon modes in the system of BEC-ed QHExs cannot be described by the NG gapless modes. As will be shown in the next section, in similar case of the BEC of coplanar 2D magnetoexcitons the optical plasmon branch is also the unique NG mode. The exciton branches (energy and quasienergy) of the spectrum have the gaps in the point $k = 0$, the roton-type behavior at intermediary values of the wave vectors and a saturation dependences at $k \to \infty$. At the same time the acoustical plasmon branch in the case of magnetoexcitons in the range of small wave vectors reveals the absolute instability. Its values are pure imaginary. In the case of BEC of magnetoexcitons there is one NG optical plasmon mode, two gapped exciton modes and one unstable acoustical plasmon mode. This agrees qualitatively with the results obtained by Fertig[81] in the case of BEC of QHExs where a single gapless NG mode of the optical plasmon type was revealed while the other modes of the spectrum were not identified at infinitesimal energies.

11. TRUE, QUASI AND UNSTABLE NAMBU-GOLDSTONE MODES OF THE BOSE-EINSTEIN CONDENSED COPLANAR MAGNETOEXCITONS

In this section we will present the results following the Refs. [30, 31] for the energy spectrum of the collective elementary excitations arising above the ground state of the Bose–Einstein condensed coplanar magnetoexcitons.

The full Hamiltonian describing the interaction of electrons and holes lying on the LLLs is:

$$H = H_{\text{Coul}} + H_{\text{Suppl}} \tag{146}$$

Where H_{Coul} is the Hamiltonian of the Coulomb interaction of the electrons and holes lying on their LLLs:

$$\hat{H}_{\text{Coul}} = \frac{1}{2}\sum_{\vec{Q}} W_{\vec{Q}}[\hat{\rho}(\vec{Q})\hat{\rho}(-\vec{Q}) - \hat{N}_e - \hat{N}_h] - \mu_e\hat{N}_e - \mu_h\hat{N}_h \tag{147}$$

and H_{Suppl} is the supplementary indirect interactions between electrons and holes, which appear due to the simultaneous virtual quantum transitions of two particles from the LLLs to excited Landau levels (ELLs) and their return back during the Coulomb scattering processes. The expression for this interaction was obtained in Ref. [25] and has the form:

$$H_{\text{suppl}} = \frac{1}{2}B_{i-i}\hat{N} - \frac{1}{4N}\sum_{Q} V(Q)\hat{\rho}(\vec{Q})\hat{\rho}(-\vec{Q}) - \frac{1}{4N}\sum_{Q} U(Q)\hat{D}(\vec{Q})\hat{D}(-\vec{Q}) \tag{148}$$

Here $\hat{\rho}(\vec{Q})$ are the density fluctuation operators expressed through the electron $\hat{\rho}_e(\vec{Q})$ and hole $\hat{\rho}_h(\vec{Q})$ density operators as follows:

$$\begin{gathered}
\hat{\rho}_e(\vec{Q}) = \textstyle\sum_t e^{iQ_y t l^2} a_{t-Qx/2} a_{t-Qx/2} \\
\hat{\rho}_h(\vec{Q}) = \textstyle\sum_t e^{iQ_y t l^2} b^{\dagger}_{t-Qx/2} b_{t-Qx/2} \\
\hat{\rho}(\vec{Q}) = \hat{\rho}_e(\vec{Q}) - \hat{\rho}_h - (\vec{Q}); \quad \hat{D}_e\vec{Q} = \hat{\rho}_e(\vec{Q}) + \hat{\rho}_h - (\vec{Q}) \\
\hat{N}_e = \hat{\rho}_e(0); \quad \hat{N}_h = \hat{\rho}_h(0) \\
\hat{N} = \hat{N}_e + \hat{N}_h W_{(\vec{Q})} = \frac{2\pi e^2}{\varepsilon_0 S|\vec{Q}|} e^{-Q^2 l^2/2}
\end{gathered} \tag{149}$$

The density operators are integral two-particle operators. They are expressed through the single-particle creation and annihilation operators $a_p^{\dagger}, a_p$ for electrons and $b_p^{\dagger}$, b_p for holes. Here, ε_0 is the dielectric constant of the background; μ_e and μ_h are chemical potentials for electrons and holes, and coefficients $V(Q)$, $U(Q)$ and B_{i-i} were calculated in Refs. [25, 29].

The starting Hamiltonian (146) has two continuous symmetries. One is the gauge global symmetry $U(1)$ and another one is the rotational symmetry $SO(2)$, so that the total symmetry is $U(1)\times SO(2)$. The gauge symmetry is generated by the operator $\hat{N}$ of the full particle number, when it commutes with the Hamiltonian. It means that the Hamiltonian is invariant under the unitary transformation $\hat{U}(\varphi)$ as follows

$$\hat{U}(\varphi)\hat{H}\hat{U}^{-1}(\varphi) = \hat{H}; \quad \hat{U}(\varphi) = e^{i\hat{N}\varphi}; \quad [\hat{H}, \hat{N}] = 0 \tag{150}$$

The operator $\hat{N}$ is referred to as the symmetry generator. The rotational symmetry $SO(2)$ is generated by the rotation operator $\hat{C}_z(\varphi)$ which rotates the in-plane wave vectors $\vec{Q}$ on the arbitrary angle φ around z axis, which is perpendicular to the layer plane and is parallel to the external magnetic field. Coefficients $W_{\vec{Q}}$, $U(\vec{Q})$ and $V(\vec{Q})$ in formulas (6) and (9) of Ref. [30] depend on the square wave vector $\vec{Q}$ which is invariant under the rotations $\hat{C}_z(\varphi)$. This fact determines the symmetry $SO(2)$ of the Hamiltonian (146). The gauge symmetry of Hamiltonian (146) after the phase transition to the Bose–Einstein condensation (BEC) state is broken as it follows from expression (16) of Ref. [30]. In terms of the Bogoliubov theory of quasiaverages, it contains a supplementary term proportional to $\tilde{\eta}$. The gauge symmetry is broken because this term does not commute with operator $\hat{N}$. Moreover, this term is not invariant under the rotations $\hat{C}_z(\varphi)$, because the in-plane wave vector $\vec{k}$ of the BEC is transformed into another wave vector rotated by the angle φ in comparison with the initial position. The second continuous symmetry is also broken. Thus, the installation of the Bose–Einstein condensation state with arbitrary in-plane wave vector $\vec{k}$ leads to the spontaneous breaking of both continuous symmetries.

We will consider a more general case of $\vec{k}\neq 0$ taking the case $\vec{k}=0$ as a limit $\vec{k}\to 0$ of the cases with small values $kl << 1$. One should keep in mind that the supplementary terms in Hamiltonian (146) describing influence of the ELLs are actual in the range of small value $kl < 0.5$. Above we established that the number of the broken generators (BGs) denoted as N_{BG} equals to two ($N_{BG}=2$).

As discussed in previous papers,[22–25, 27, 76, 91, 92] the breaking of the gauge symmetry of the Hamiltonian (146) can be achieved using the Keldysh–Kozlov–Kopaev[93] method with the unitary transformation

$$\hat{D}(\sqrt{N_{ex}}) = \exp[\sqrt{N_{ex}}(d^{\dagger}(\vec{k}) - d(\vec{k}))] \tag{151}$$

where $d^{\dagger}(\vec{k})$ and $d(\vec{k})$ are the creation and annihilation operators of the magnetoexcitons. In the electron–hole representation they are:[22–25, 27, 76, 91, 92]

$$d^\dagger(\vec{P}) = \frac{1}{\sqrt{N}}\sum_t e^{-iP_y t l^2} a^\dagger_{t+P_x/2} b^\dagger_{-t+P_x/2}$$
$$d(\vec{P}) = \frac{1}{\sqrt{N}}\sum_t e^{iP_y t l^2} b_{-t+P_x/2} a_{t+P_x/2} \tag{152}$$

BEC of the magnetoexcitons leads to the formation of a coherent macroscopic state as a ground state of the system with wave function

$$|\psi_g(\vec{k})\rangle = \hat{D}(\sqrt{N_{ex}})|0\rangle;\ a_p|0\rangle = b_p|0\rangle = 0 \tag{153}$$

Here $|0\rangle$ is the vacuum state for electrons and holes. In spite of the fact that we kept arbitrary value of $\vec{k}$, nevertheless our main goal is the BEC with $\vec{k}=0$ and we will consider the interval $0.5 > kl \geq 0$. The function (153) will be used to calculate the averages values of the type $\langle D(\vec{Q})D(-\vec{Q})\rangle$. The transformed Hamiltonian (146) looks like:

$$\hat{\mathscr{H}} = D(\sqrt{N_{ex}})HD^\dagger(\sqrt{N_{ex}}) \tag{154}$$

and is succeeded, as usual, by the Bogoliubov u–v transformations of the single-particle Fermi operators

$$\alpha_p = \hat{D}(\sqrt{N_{ex}})a_p\hat{D}^\dagger(\sqrt{N_{ex}}) = ua_p - v\left(p-\frac{k_x}{2}\right)b^\dagger_{k_x-p};$$
$$\alpha_p|\psi_g(\vec{k})\rangle = 0$$
$$\beta_p = \hat{D}(\sqrt{N_{ex}})b_p\hat{D}^\dagger(\sqrt{N_{ex}}) = ub_p + v\left(\frac{k_x}{2}-p\right)a^\dagger_{k_x-p};$$
$$\beta_p|\psi_g(\vec{k})\rangle = 0 \tag{155}$$

Instead of this traditional way of transforming the expressions of the starting Hamiltonian (146) and of the integral two-particle operators (149) and (152), we will use the method proposed by Bogoliubov in his theory of quasiaverages,[1, 51] remaining in the framework of the original operators. The new variant is completely equivalent to the previous one, and both of them can be used in different stages of the calculations. For example, the average values can be calculated using the wave function (153) and u-v transformations (155), whereas the equations of motion for the integral two-particle operators can be simply written in the starting representation.

The Hamiltonian (146) with the broken gauge symmetry in the lowest approximation has the form

$$\begin{aligned}\hat{\mathscr{H}} = {} & \frac{1}{2}\sum_{\vec{Q}} W_{\vec{Q}}[\rho(\vec{Q})\rho(-\vec{Q}) - \hat{N}_e - \hat{N}_h] - \mu_e\hat{N}_e - \mu_h\hat{N}_h \\ & + \frac{1}{2}B_{i-i}\hat{N} - \frac{1}{4N}\sum_Q V(Q)\hat{\rho}(\vec{Q})\hat{\rho}(-\vec{Q}) - \frac{1}{4N} \\ & \times \sum_Q U(Q)\hat{D}(\vec{Q})\hat{D}(-\vec{Q}) \\ & - \tilde{\eta}\sqrt{N}(d^\dagger(k) + d(k))\end{aligned} \tag{156}$$

For simplicity another smaller term of this type proportional to $\tilde{\eta}$ was dropped. Here parameter $\tilde{\eta}$, which determines the breaking of the gauge symmetry, depends on the chemical potential μ and on the square root of the density, similar to the case of weakly non-ideal Bose-gas considered by Bogoliubov.[1, 51] In our case the density is proportional to the filling factor $\nu = v^2$ and we have:

$$\begin{aligned}&\mu = \mu_e + \mu_h;\quad \bar{\mu} = \mu + I_l;\quad N_{ex} = v^2N;\\ &\tilde{E}_{ex}(k) = -I_l - \Delta(k) + E(k)\\ &\tilde{\eta} = (\tilde{E}_{ex}(k) - \mu)v = (E(k) - \Delta(k) - \bar{\mu})v;\\ &E(k) = 2\sum_Q W_Q \mathrm{Sin}^2\left(\frac{[K\times Q]_z l^2}{2}\right)\end{aligned} \tag{157}$$

The equations of motion for the integral two-particle operators with wave vectors $\vec{P}\neq 0$ in the special case of BEC of magnetoexcitons with $\vec{k}=0$ are

$$\begin{aligned}i\hbar\frac{d}{dt}d(\vec{P}) & \\ = {} & [d(\vec{P}), \hat{\mathscr{H}}] = (-\bar{\mu} + E(\vec{P}) - \Delta(\vec{P}))d(\vec{P}) \\ & - 2i\sum_{\vec{Q}}\tilde{W}(\vec{Q})Sin\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right)\hat{\rho}(\vec{Q})d(\vec{P}-\vec{Q}) \\ & - \frac{1}{N}\sum_{\vec{Q}}U(\vec{Q})\mathrm{Cos}\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right) \\ & \times D(\vec{Q})d(\vec{P}-\vec{Q}) + \tilde{\eta}\frac{D(\vec{P})}{\sqrt{N}} \\ i\hbar\frac{d}{dt}d^\dagger(-\vec{P}) & \\ = {} & [d^\dagger(-\vec{P}), \hat{\mathscr{H}}] = (\bar{\mu} - E(-\vec{P}) + \Delta(-\vec{P}))d^\dagger(-\vec{P}) \\ & + 2i\sum_{\vec{Q}}\tilde{W}(\vec{Q})\mathrm{Sin}\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right)d^\dagger(-\vec{P}-\vec{Q})\hat{\rho}(-\vec{Q}) \\ & + \frac{1}{N}\sum_{\vec{Q}}U(\vec{Q})\mathrm{Cos}\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right)d^\dagger(-\vec{P}-\vec{Q}) \\ & \times D(-\vec{Q}) - \tilde{\eta}\frac{D(\vec{P})}{\sqrt{N}} \\ i\hbar\frac{d}{dt}\hat{\rho}(\vec{P}) & \\ = {} & [\hat{\rho}(\vec{P}), \hat{\mathscr{H}}] = -i\sum_{\vec{Q}}\tilde{W}(\vec{Q})\mathrm{Sin}\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right) \\ & \times[\hat{\rho}(\vec{P}-\vec{Q})\hat{\rho}(\vec{Q}) + \hat{\rho}(\vec{Q})\hat{\rho}(\vec{P}-\vec{Q})] \\ & + \frac{i}{2N}\sum_{\vec{Q}}U(\vec{Q})\mathrm{Sin}\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right) \\ & \times[D(\vec{P}-\vec{Q})D(\vec{Q}) + D(\vec{Q})D(\vec{P}-\vec{Q})]\end{aligned} \tag{158}$$

$$
\begin{aligned}
& i\hbar\frac{d}{dt}\hat{D}(\vec{P}) \\
&\quad = [\hat{D}(\vec{P}), \hat{\mathscr{H}}] - i\sum_{\vec{Q}} \tilde{W}(\vec{Q})\mathrm{Sin}\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right) \\
&\qquad \times[\hat{\rho}(\vec{Q})\hat{D}(\vec{P}-\vec{Q}) + \hat{D}(\vec{P}-\vec{Q})\hat{\rho}(\vec{Q})] \\
&\qquad + \frac{i}{2N}\sum_{\vec{Q}} U(\vec{Q})\mathrm{Sin}\left(\frac{[\vec{P}\times\vec{Q}]_z l^2}{2}\right) \\
&\qquad \times[\hat{D}(\vec{Q})\hat{\rho}(\vec{P}-\vec{Q}) + \hat{\rho}(\vec{P}-\vec{Q})\hat{D}(\vec{Q})] \\
&\qquad + 2\tilde{\eta}\sqrt{N}[d(\vec{P}) - d^\dagger(-\vec{P})]
\end{aligned}
$$

Following the equations of motion (158) we introduce four interconnected retarded Green's functions at $T = 0$[94, 95]

$$
\begin{aligned}
G_{11}(\vec{P}, t) &= \langle\langle d(\vec{P}, t); \hat{X}^\dagger(\vec{P}, 0)\rangle\rangle \\
G_{12}(\vec{P}, t) &= \langle\langle d^\dagger(-\vec{P}, t); \hat{X}^\dagger(\vec{P}, 0)\rangle\rangle \\
G_{13}(\vec{P}, t) &= \left\langle\left\langle \frac{\hat{\rho}(\vec{P}, t)}{\sqrt{N}}; \hat{X}^\dagger(\vec{P}, 0)\right\rangle\right\rangle \\
G_{14}(\vec{P}, t) &= \left\langle\left\langle \frac{\hat{D}(\vec{P}, t)}{\sqrt{N}}; \hat{X}^\dagger(\vec{P}, 0)\right\rangle\right\rangle
\end{aligned} \tag{159}
$$

We also need their Fourier transforms $G_{ij}(\vec{P}, \omega)$, for which the equations of motion of the type similar to the equations of motion (158) were obtained. These Green's functions can be named as one operator Green's functions, because they contain only one two-particle operator of the type $d^\dagger$, d, ρ, D. At the same time, in the right hand side of the corresponding equations of motion there is a second generation of two-operator Green's function containing the different products of the two-particle operators mentioned above. For these operators was derived the second generation of the equations of motion containing in their right sides the Green's function of the third generation. They are the three-operator Green's functions for which it is necessary to derive the third generation of equations of motion. However, we have to terminate here the evolution of the infinite chains of equations of motion for multi-operators Green's function following the procedure proposed by Zubarev.[95] The truncation of the chains of the equations of motion and the decoupling of the one-operator Green's functions from the multi-operator Green's functions was achieved substituting the three operator Green's functions by the one-operator Green's functions multiplied by the average value of remaining two operators. The average values were calculated using the ground state wave function (153) and u–v transformations (155). The Zubarev procedure is equivalent to a perturbation theory with a small parameter of the type $v^2(1-v^2)$, which represent the product of a filling factor $\nu = v^2$ and the phase-space filling factor $(1-v^2)$ reflecting the Pauli exclusion principle.

The closed system of Dyson equations has the form

$$
\sum_{j=1}^{4} G_{1j}(\vec{P}, \omega)\Sigma_{jk}(\vec{P}, \omega) = C_{1k}; \quad k = 1, 2, 3, 4 \tag{160}
$$

There are 16 different components of the self-energy parts $\Sigma_{jk}(\vec{P}, \omega)$ forming a 4×4 matrix. Due to the structure of the self-energy parts the cumbersome dispersion equation can be expressed in general form by the determinant equation

$$
\det|\Sigma_{ij}(\vec{P}, \omega)| = 0 \tag{161}
$$

It splits into two independent equations. One of them concerns only the optical plasmon branch and has a simple form

$$
\Sigma_{33}(\vec{P}, \omega) = 0 \tag{162}
$$

It does not include the chemical potential $\bar{\mu}$ and the quasi-average constant $\tilde{\eta}$. The second equation contains the self-energy parts Σ_{11}, Σ_{22}, Σ_{44}, Σ_{14}, Σ_{41}, Σ_{24} and Σ_{42}, which include both parameters $\bar{\mu}$ and $\tilde{\eta}$. The second equation reads

$$
\begin{aligned}
&\Sigma_{11}(\vec{P}; \omega)\Sigma_{22}(\vec{P}; \omega)\Sigma_{44}(\vec{P}; \omega) \\
&\quad - \Sigma_{41}(\vec{P}; \omega)\Sigma_{22}(\vec{P}; \omega)\Sigma_{14}(\vec{P}; \omega) \\
&\quad - \Sigma_{42}(\vec{P}; \omega)\Sigma_{11}(\vec{P}; \omega)\Sigma_{24}(\vec{P}; \omega) = 0
\end{aligned} \tag{163}
$$

The solution of the Eq. (162) is

$$
\begin{aligned}
(\hbar\omega(P))^2 &= \frac{\langle D(P)D(-P)\rangle}{N^2} \\
&\quad \times \sum_{Q} U(Q)(U(-Q) - U(Q-P)) \\
&\quad \times \mathrm{Sin}^2\left(\frac{[P\times Q]_z l^2}{2}\right)
\end{aligned} \tag{164}
$$

The right hand side of this expression at small values of P has the dependence $|P|^4$ and tends to saturation at large values of P. The optical plasmon branch $\hbar\omega_{OP}(P)$ has a quadratic dispersion law in the long wavelength limit and saturates in the range of short wavelengths. It depends on concentration as $\sqrt{v^2(1-v^2)}$ what coincides with the concentration dependencies of 3D plasma $\omega_p^2 = (4\pi e^2 n_e)/\varepsilon_0 m$[96] and 2D plasma $\omega_p^2(q) = (2\pi e^2 n_s q)/\varepsilon_0 m$,[88] where n_e and n_s are the corresponding density of electrons. The supplementary factor $(1-v^2)$ in our case reflects the Pauli exclusion principle and the vanishing of the plasma oscillations at $\nu = v^2 = 1$. The obtained dispersion law is shown in Figure 2. Similar dispersion law was obtained for the case of 2D electron–hole liquid (EHL) in a strong perpendicular magnetic field,[97] when the influence of the quantum vortices created by electron and hole subsystems is compensated exactly. However, the saturation dependencies in these two cases are completely different. In the case of Bose–Einstein condensed magnetoexcitons it is determined by the ELLs,

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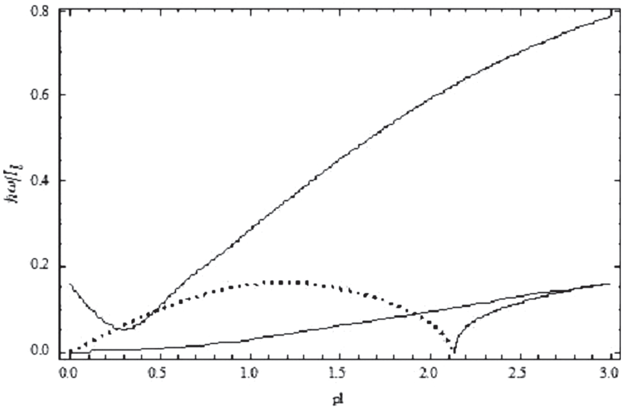

Fig. 2. Three branches of the collective elementary excitations: the exciton-type quasi-NG mode with a gap in the point $pl = 0$; the second-type NG mode describing the optical plasmons and the first-type NG mode with absolute instability (dotted line) describing the acoustical plasmons.

whereas in the case of EHL[97] it is determined by the Coulomb interaction in the frame of the LLLs.

The acoustical plasmon branch has the dispersion law, which is completely different from the optical plasmon oscillations. It has an absolute instability beginning with the small values of wave vector going on up to the considerable value $pl \approx 2$. In this range of wave vectors, the optical plasmons have energies which do not exceed the activation energy $U(0)$. It means that the optical plasmons containing the opposite-phase oscillations of the electron and hole subsystems without displacement as a whole of their center of mass are allowed in the context of the attractive bath generated by the ELLs. On the other hand, the in-phase oscillations of the electron and hole subsystems in the composition of the acoustical plasmons are related to the displacements of their center of mass. Such displacements can take place only if their energy exceeds the activation energy $U(P)$ existing due the attractive bath. As a result, the acoustical plasmon branch has an imaginary part represented by the dashed line and is completely unstable in the region of wave vectors $pl \leq 2$. At greater values $pl > 2$ the energy spectrum is real and nonzero, approaching to the energy spectrum of the optical plasmons.

In case of 2D magnetoexcitons in the BEC state with small wave vector $kl < 0.5$ described by the Hamiltonian (156), one should take into account that both continuous symmetries usual for the initial form (146) are lost. It happened due to the presence of the term $\tilde{\eta}(d^{\dagger}_{\vec{k}} + d_{\vec{k}})$ in the frame of the Bogoliubov theory of the quasiaverages. Nevertheless the energy of the ground state as well as the self-energy parts $\Sigma_{ij}(P, \omega)$ were calculated only in the simplest case of the condensate wave vector $\vec{k} = 0$. These expressions can be relevant also for infinitesimal values of the modulus $|\vec{k}|$ but with a well defined direction. In this case the symmetry of the ground state will be higher than that of the Hamiltonian (156), what coincides with the situation described by Georgi and Pais.[33] It is one possible explanation of the quasi-NG modes appearance in the case of exciton branches of the spectrum. Another possible mechanism of the gapped modes appearance is the existence of the local gauge symmetry, the breaking of which leads to the Higgs effect.[7] The interaction of the electrons with the attached vortices gives rise to a gapped energy spectrum of the collective elementary excitations as was established in Refs. [42, 54]. The number of the NG modes in the system with many broken continuous symmetries was determined by the Nielsen and Chadha[17] theorem. It states that the number of the first-type NG modes N_I being accounted once and the number of the second type NG modes N_{II} being accounted twice equals or prevails the number of broken generators N_{BG}. It looks as follows $N_I + 2N_{II} \geq N_{BG}$. In our case the optical plasmon branch has the properties of the second-type NG modes. We have $N_I = 0$; $N_{II} = 1$ and $N_{BG} = 2$. It leads to the equality $2N_{II} = N_{BG}$. The three branches of the energy spectrum are represented together in the Figure 2. One of then is a second-type Nambu-Goldstone (NG) mode describing the optical plasmon-type excitations, the second branch is the first-type NG mode with absolute instability describing the acoustical-type excitations and the third branch is the quasi-NG mode describing the exciton-type collective elementary excitations of the system.

We can repeat that results obtained in the magnetoexciton system are similar to those obtained for the system of BEC of the quantum Hall excitons (QHExs).[81] In these both models there is only one gapless Nambu-Goldstone mode between four branches of the energy spectrum. In our model it is related with the optical plasmon branch, whereas in the case of QHExs this mode is represented by the superposition of the operators describing the optical plasmon and exciton modes. In both models the exciton branches of the spectrum are not gapless and differ from the NG modes. In our case the exciton energy and quasienergy branches corresponding to normal and abnormal Green's functions have a gaps in the point $p = 0$, a roton-type segments in the region of intermediary wave vectors $pl \sim 1$ and saturation-type behaviors at great values of pl. In the case of Ref. [81] the exciton type response function $\chi_z(q, \omega)$ and the acoustical-type response function $\chi_F(q, \omega)$ have no poles in the region of small energies in the frame of the LLLs. It was concluded that the energies of these excitations may be situated at greater values. In our case the acoustical plasmon branch reveals an absolute instability in the range of small and intermediary values of p. It means that a real values of the pole does not exist in the range of small energies, which is similar with the results of Ref. [81]. One can conclude that the qualitative properties of the energy spectra in both models are similar in spite of the mentioned differences. It is an additional argument supporting the accuracy of our calculations, which satisfy to the Nielsen and Chadha theorem.[17]

The result concerning the BEC at $T = 0$ are estimates which describe the real situation at finite temperatures lower than the critical temperature of the Berezinskii–Kosterlitz–Thouless (BKT) topological phase transition[79, 80] related with the existence of the vortices and their clusters such as bound vortex-antivortex pairs. Just the unbinding of these pairs determines the critical temperature $T_{BKT} = \pi n\hbar^2/2mk_B$, where n is the surface density of the Bose particles and m is their mass. On one side of the phase transition there is a quasi-ordered fluid and on the other is a disordered unbounded vortex plasma. Although the formation of an isolated vortex will not occur at low temperature due its extensive creation energy, there always can be production of a pair of vortices with equal and opposite charges since the perturbation produced by such a pair falls off sufficiently rapidly at large distances so that their energy is finite.[80] Such topological formations can be easily created by the thermal fluctuations.

The presence of the vortex clusters makes the previously infinite homogeneous 2D e–h system to become nonhomogeneous as a whole. However, the local homogeneity with finite local surface areas can exist leading to the BEC with finite critical temperature $T_c = 2\pi n\hbar^2/mk_B \lg(nS)$.[98] Instead of an off diagonal long-range order as in the case of 3D Bose gas in the 2D systems there is only a long rang correlations, which decays algebraically with distance. In such a way the quantum vortices promote the BEC and the formation of the superfluid component of the 2D Bose-gas at finite temperatures and at the same time the superfluid component is necessary for the formation of the quantum vortices. It is a self-organization-type situation. The BKT phase transition is a widely studied phenomenon.[99–101]

Attempts to discover experimentally the spontaneous symmetry breaking in the exciton range of the spectrum and the efforts to evidence the spontaneous coherence in the 2D excitonic systems will be considered in the next section on the basis of the Refs. [102–146].

12. SPONTANEOUS COHERENCE IN 2D EXCITONIC SYSTEMS

As was mentioned by Snoke in Refs. [102, 103] recent experimental efforts of several groups have demonstrated the spontaneous coherence in polariton systems, which can be viewed as a type of nonequilibrium BEC. The system of polaritons in the quantum wells embedded into the microcavity reveals the phenomenon of BEC and superfluity. The achievements in this field are presented in Refs. [104–110]. In these systems the polariton lifetime is longer than, but not much longer than the polariton–polariton scattering time, which leads to the thermalization. By contrast over past twenty years several groups of investigators represented by Snoke,[111–121] Butov,[122–130] Timofeev,[131–138] Krivolapchuk,[139–143] Fukuzawa[144–146] and their coworkers have pursued experiments in double quantum well (DQW) excitonic systems with very long lifetime. In these systems the indirect excitons (IXs) formed from spatially separated electrons and holes have dipole moments oriented perpendicularly to the layers. They are named dipole excitons and their interaction is not a short-range contact interaction but instead a long-range dipole–dipole repulsion. We briefly recall the results obtained in Refs. [142, 143].

When analyzing the possibility of BEC in a 2D system it should be noted that at $T \neq 0$ condensation of a homogeneous 2D gas is impossible because of destruction of the condensate by thermal fluctuations.[147] In a 2D system $\rho(E)$ is constant and the integral $N = \sum_k N_k = \int_0^\infty \rho(E)dE/e^{(E-\mu)/k_BT} - 1$ would diverge at $\mu \to 0$ and $T \neq 0$ because of the zero denominator at the lower integrating limit and therefore BEC is impossible here. Physically this fact means that the maximal occupation of free states ($E > 0$) is infinite. However, if in a 2D boson system, together with free excitons, there are present some discrete states (localized states whose existence is caused by the appearance of fluctuations of the heterointerface potential[148]) ε_0, ε_1 etc, such that $\varepsilon_0 < \varepsilon_1 < E = 0$, the situation changes essentially. In this case under increasing number of bosons in the system the value of chemical potential cannot be arbitrarily close to the $E = 0$ value because of the $N(\varepsilon_0) \geq 0$ requirement, so $(-\mu)_{\min} = |\varepsilon_0|$ and, consequently, the integral N has a finite value:[149]

$$n_c(T) = -\frac{mk_BT}{2\pi\hbar} \ln(1 - \exp^{-|\varepsilon_0|/k_BT}) \tag{165}$$

Therefore, at the moment when n exceeds $n_c(T)$, localized states are occupied by a macroscopic number of particles: $n - n_c(T) = n(\varepsilon_0) + n(\varepsilon_1)$. This means that BEC into localized states occurs in a limited space region. In this sense BEC in a system of 2D bosons, having a discrete spectrum of energy together with a continuous one, resembles the experimentally discovered phenomenon of BEC in atoms of alkali metals in space-limited traps produced by a magnetic field.[150]

The obvious advantage of an IX in a DQW as a perspective system to reveal BEC is the possibility of controlling effectively its radiative lifetime τ_R with the help of external effects. So, for example, an electric field V_{dc} applied to a DQW in the direction of the growth axis of the structure causes an essential decrease of the overlap of wave-functions of an electron and a hole in an IX in the z direction and, as a result, τ_R increases significantly (by up to three orders of magnitude[145]). This allows a more effective cooling of the system to the bath temperature and, of equal importance, gives an opportunity to increase the concentration of the IX gas without increasing the pumping density. The latter circumstance plays an important role in the experiment since it allows us to decrease heating of the sample by phonons that are inevitably radiated at relaxation of photoexcited carriers and excitons. Just this heating of the sample under investigation is often the main cause of the impossibility of reaching the critical temperature of the boson gas in experiments that use large optical pumping densities to create the critical density of bosons having very short lifetimes.

Convincing evidence of the BEC effect would be an appropriate exciton distribution function over energies (momenta) obtained in an experiment. In general the exciton distribution function can be determined in experiment by the form of the phonon replica line in luminescence spectra, but in the performed study no phonon replica was observed. This was why in this paper the nonphonon luminescence line of space IXs was studied. However, since the intensity of exciton radiation is proportional to the occupation of radiative states by particles, it indirectly reflects the distribution function of excitons over the free and localized states, which both contribute to the formation of the inhomogeneously broadened IX line. Due to this fact one can hope that studies of IX luminescence will reveal the BEC effect predicted[149] for a system of 2D bosons which are distributed over the free and localized states.

A giant (threefold) increase of luminescence intensity of a part of the spectral profile of the IX line in DQWs of $GaAs/Al_{0.33}Ga_{0.67}As$ on changing the temperature of the sample and the value of the external electric field applied to DQWs was discovered. Besides that, the luminescence intensity of this part of the spectral profile of the IX line fluctuated with the characteristic time of tens of seconds. Such an unusual behaviour of the IX line was regarded as possible evidence for BEC in a system of 2D bosons placed in fluctuations of potential formed by heterointerfaces of the sample.

Figure 1 of the Ref. [142] is reproduced in Figure 3. It shows the luminescence spectra of the DQW, dependent on V_{dc} at $T = 1.8$ K and the density of optical excitation $P = 5$ W cm^{-2}. Here, at $V_{dc} = 0$ (figure (a)) the radiation spectrum was close to that of the flat-band case and it consisted of two lines, DXW and DXN, corresponding to luminescence of direct excitons (DXs) from the wide and the narrow wells respectively. At nonzero V_{dc} (figures (b)–(e)) an indirect regime was achieved (see the inset) when the IX line took the lowest energetic position in the PL spectra. When V_{dc} increased the IX line moved monotonically towards lower energies.

It was noted that in some interval of V_{dc} a giant (up to threefold) increase (shot) of intensity of a part of the IX spectral line profile (figure (d)) Ref. [142] occurred. A very important circumstance was that the intensity shot was absent from the whole investigated interval of spectral positions of the IX line at any temperature $4.2\ \text{K} \leq T \leq 30$ K and optical pumping densities of $P \leq 5$ W cm^{-2}. Thus the spectral profile of the IX line corresponding to the case of figure (d) was shown on a large scale in the Figure 3 of Ref. [142] and it is reproduced here in Figure 4. It has some interesting peculiarities consisting of a narrow intense line C and of 'wings' W having significantly smaller intensity. Measurements of the temporal

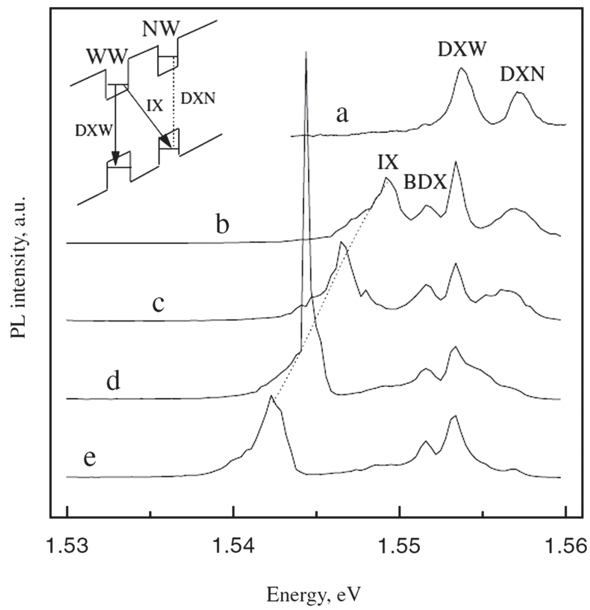

Fig. 3. PL spectra taken at $T = 1.8$ K, $P = 5$ W cm^{-2} and $V_{dc} = 0$ V (a), -0.5 V (b), -2 V (c). The inset shows the indirect regime of the DQW following the Ref. [142]. Reprinted with permission from [142], V. V. Krivolapchuk et al., *Nanotechnology* 11, 246 (**2000**). © 2000, IOP Publishing.

evolution of intensities of the C and W components have shown that the intensity of C, in contrast to that of W, fluctuated in time (changing threefold) on a characteristic scale of tens of seconds.

The shape of the IX luminescence line in a DQW was inhomogeneous and was determined by exciton radiation from different space regions of the DQWs plane, which differed from one another by the thickness of the QW layers, by fluctuations of barrier composition and by the value of the local electric field of impurities. The emission

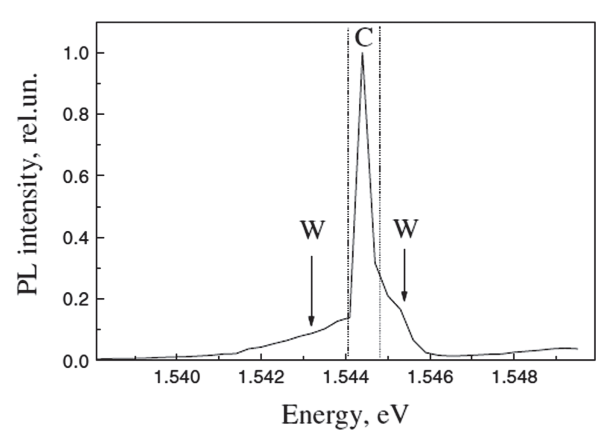

Fig. 4. IX line spectral profile of figure (d). Two vertical dotted lines separate different spectral parts of the IX line profile which shown (C) and did not show (W) temporal evolution of the PL intensity. The data were obtained in Ref. [142]. Reprinted with permission from [142], V. V. Krivolapchuk et al., *Nanotechnology* 11, 246 (**2000**). © 2000, IOP Publishing.

intensity of each spectral fragment of the IX line was proportional to the exciton occupation of the corresponding space region in the plane of the QW. However, a spectral region (V_{dc}) where the situation changes essentially at $T = 1.8$ K and $P = 5$ W cm^{-2} appeared. In this region, shown in Figure 3(d) a significant (threefold) increase of luminescence intensity of a part of the spectral profile of the IX line and a consequent increase (1.5-fold) of integral intensity I_{IX} had been observed. Such a behavior was anomalous in comparison to the monotonic decrease of I_{IX} (and, most important, to the absence of the intensity shot) with increasing V_{dc} under other experimental conditions (at $T = 1.8$ K and $P < 1$ W cm^{-2}; 4.2 K $\leq T \leq$ 30 K and any $P \leq 5$ W cm^{-2} as well). This anomalous behavior of the IX line indicated that in the case of Figure 1(d) of Ref. [142] there are much more particles participating in radiation (occupying states which can radiate) than in the cases of Figures 1(b), (c), (e) of Ref. [142].

To explain this the BEC model developed for 2D systems[149] was proposed. It was shown that if in a system of bosons (excitons) there is a localized state ε_0 below the bottom of the exciton band, the chemical potential of excitons is trapped by the localized level ε_0 and, as a consequence, the number n_c of particles appears to be finite. Therefore just when the concentration n of excitons in the system under consideration exceeds n_c, a macroscopic number $n - n_c$ of particles comes to the lowest energetic state of the whole boson system (i.e., free zone and the localized state) and that leads to the appearance of a condensate.

The phenomenon of a giant intensity shot was revealed in an experiment when the IX line shifted to lower energies (that means changing V_{dc}), this being a consequence of an increase of the radiative lifetime of excitons due both to an increase of the exciton concentration (at constant optical pumping) and to their effective thermalization to the bath temperature. These two circumstances according to the authors opinion caused the BEC that leaded to a huge population of a localized state.

Since, as noted above, all localized excitons take part in radiative recombination, this resulted in a significant change of the shape of the IX luminescence line from the case of figure (c) to that of figure (d).

Thus the totality of the experimental data describing the evolution of the IX luminescence line (namely, a giant rise of intensity of a part of the spectral profile of the IX line followed by long-time oscillations) in the authors opinion provided evidence that in a system of IXs of high density a Bose–Einstein condensate at localized states (traps) in DQWs appeared. Thermal equilibrium of this type of excitons in a trap has been demonstrated experimentally.[102, 103]

As was appreciated in the Refs. [102, 103] up to now there has not been an universally accepted demonstration of BEC in this type of systems and is necessary a better understanding of the many-body effects of the interacting dipole IXs. But the accumulated knowledge permits to formulate some conclusions. One of them states that the confinement of the excitons in a trap analogous to the optical traps for the cold atoms is a great advantage instead of creating excitons with a laser allowing them to expand freely out of the excitation region. Recent work[117] showed that the IXs in DQWs reach equilibrium both energetically and spatially in a stress induced trap. One variant of the BEC of excitons equivalent to the BEC in a trap was proposed by Jan and Lee[149] and was used in Refs. [142, 143]. Another conclusion formulated in Refs. [102, 103] concerns the role of the temperature. If the temperature is low compared to the energy fluctuation due to the disorder then the excitons will become trapped in low energy minima of the disorder potential and will not act as a free gas. Such energy minima can localize only one or a small number of IXs because they are repealing each other due to the dipole–dipole interaction. In difference on it in the trap there are localized levels able to accommodate a macroscopic number of IXs. A strong energy shift due to interactions may cancel out the trapping potential and may flatten it.[121]

Another important conclusion of Refs. [102, 103] is based on the results by Laikhtman and Rapaport[151] who underlined that the dipole IXs in a coupled quantum wells (CQWs) no longer act as a gas but rather as a correlated liquid. This does not mean that BEC is impossible at high density. But the canonical telltale for condensation in a weakly interacting Bose gas, namely a peak of occupation number at $k = 0$ may not be easily seen in these systems. It would be better to look for hydrodynamic effects of condensation of excitons such as quantum vortices or superfluidity similar to the liquid helium.

As one can see the most eminent achievements discussed in the review article concern the famous FQHEs discovered in the frame of the one-component 2DEG. They suggest to search similar phenomena in the frame of the two-component, 2D e–h systems, when the CPs will be formed by electrons and holes with attached point vortices in different combinations taking into account the interactions inside the CPs from one side and the Coulomb interaction between the electrons and holes forming the usual magnetoexcitons from the other side. In spite of the fact that the supplementary gauge magnetic fields created by electrons and holes with opposite charges will be compensated in the mean field approximation, nevertheless the new electron quantum states will appear as elementary excitations and quantum fluctuations in these new conditions.

13. CONCLUSIONS

The purpose of the present review is to discuss the phenomena related to the spontaneous breaking due to the quantum fluctuations of the continuous symmetries existing in the frame of the two-dimensional e–h systems in a strong perpendicular magnetic field with electrons and holes lying on the lowest Landau levels. The spontaneous symmetry breaking leads to the formation of the new ground states and phase transitions and determines the energy spectra of the collective elementary excitations appearing over the new ground states.

The main attention is given to the electron–hole systems forming the coplanar magnetoexcitons in the Bose–Einstein condensation ground state with wave vector $\vec{k}=0$ under the influence of the excited Landau levels when the exciton-type excitations coexist with the plasmon-type oscillations. At the same time the properties of the 2DEG under the conditions of the FQHE as well as of the similar 2DHG spatially separated on the layers of the DQW are taken into account, so as to foresee their possible influence on the states of the coplanar magnetoexcitons when the distance between the DQW layers diminishes. Side by side with the 2DEG and 2DHG a bilayer electron systems in the conditions of the FQHE with one half filling factor of LLLs in each layer and with the total filling factor of two layers equal to unity are taken into account because the coherent superposition of the electron states in two layers happens to be equivalent with the formation of the QHExs in the coherent macroscopical state, which can be compared with the BEC of the coplanar magnetoexcitons. The breaking of the global gauge symmetry as well as of the continuous rotational symmetries leads to the formation of the gapless Nambu-Goldstone modes of the collective excitations above the selected ground state, corresponding to the macroscopical wave function with a fixed phase, whereas the breaking of the local gauge symmetry gives rise to the Higgs phenomenon characterized by the gapped branches of the energy spectrum of the collective elementary excitations. The existence of the gapless and gapped branches of the energy spectrum is equivalent to the appearance of the massless and massive particles correspondingly in the relativistic physics.

Application of the Nielsen-Chadha theorem establishing the relation between the number of the NG modes and the number of the broken symmetry operators as well as the elucidation of the conditions when the quasi-NG modes appear was effectuated on the concrete example of the spinor atoms in the state of BEC in an optical trap with the aim to better understand the results concerning the coplanar magnetoexcitons. The Higgs phenomenon gives rise to the formation of the composite particles in the frame of the 2DEG in conditions of the FQHE, so that the electron with an odd or even number of the attached point vortices behaves as an composite boson or fermion correspondingly. Their description in the frame of the Ginzburg-Landau theory is demonstrated.

Side by side with the 2D coplanar magnetoexcitons the conditions under which the spontaneous coherence may appear in the system of indirect excitations in the double quantum well structures with spatially separated electrons and holes were discussed. The experimental attempts to achieve the BEC of IXs in the traps arising due to the interface width fluctuations or due to the applied stress were reviewed, the concluding remarks and the recommendations are mentioned. The formation of the high density 2D magnetoexcitons and magnetoexciton-polaritons with point quantum vortices attached is suggested.

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